

Graph Expansions of Semigroups

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Abstract

We construct a graph expansion from a semigroup with a given generating set, thereby generalizing the graph expansion for groups introduced by Margolis and Meakin. We then describe structural properties of this expansion. The semigroup graph expansion is itself a semigroup and there is a map onto the original semigroup. This construction preserves many features of the original semigroup including the presence of idempotent/periodic elements, maximal group images (if the initial semigroup is E -dense), finiteness, and finite subgroup structure. We provide necessary and sufficient graphical criteria to determine if elements are idempotent, regular, periodic, or related by Green's relations. We also examine the relationship between the semigroup graph expansion and other expansions, namely the Birget and Rhodes right prefix expansion and the monoid graph expansion.

If S is a Σ -generated semigroup, its graph expansion is generally not Σ -generated. For this reason, we introduce a second construction, the path expansion of a semigroup. We show that it is a Σ -generated subsemigroup of the semigroup graph expansion. The semigroup path expansion possesses most of the properties of the semigroup graph expansion. Additionally, we show that the path expansion construction plays an analogous role with respect to the right prefix expansion of semigroups that the group graph expansion plays with respect to the right prefix expansion of groups.

Dedication

In memory of my mother, Catherine Noonan.

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Chapter 1

Introduction

The study of expansions has enriched the general theory of semigroups over the last thirty years. Birget and Rhodes introduced the concept of a semigroup expansion in [1] as a functor from one category of semigroups to a larger category. Informally, they describe an expansion as follows:

An expansion is, informally speaking, a systematic way of writing semigroups S as homomorphic images of other semigroups \overline{S} that have nicer properties; moreover the homomorphism $\overline{S} \twoheadrightarrow S$ should be such that some important properties of S are preserved in \overline{S} . [2]

In [1], Birget and Rhodes introduce many different semigroup expansions including the free expansion, the Rhodes expansion (they call it the machine expansion), the Henckell expansion, the prefix expansion, and the prefix expansion cut down to generators. Their goal is to use techniques from finite semigroup theory to study infinite semigroups. Birget and Rhodes further develop these constructions in [2]. In particular, they show that if starting with a group that has an interesting Burnside problem, performing the expansion produces a semigroup which also has interesting properties.

The Rhodes expansion has been useful in proving various structural results about semigroups. In particular, it has been used frequently in the proofs of decomposition theorems. Given a semigroup S , the main idea of a decomposition theorem is to find

a special product, for example a wreath product, that is a cover of S . It is key that the component semigroups of the product are related to S and have “nicer” properties than S . Tilson uses the Rhodes expansion in [22] to prove the Ideal Theorem. His result bounds how complex the decomposition of a semigroup S can be using the complexities of an ideal of S and its quotient. Henckell, Lazarus, and Rhodes employ the Rhodes expansion in [10] to prove the Holonomy Theorem. The main ingredient of the Holonomy Theorem is a monoid S with a special length function (in [10], this length function bounds the length of \mathcal{J} -chains in S). It turns out that the Rhodes expansion of S (with identity added) is isomorphic to the special product obtained in the Holonomy decomposition. They show that it has an interesting action on certain infinite rooted trees. In [20], Rhodes gives an alternative proof of the Holonomy Theorem using the Rhodes expansion.

On a different note, Le Saec, Pin, and Weil use the Rhodes expansion in [16] to construct an automaton which is then used to form a new semigroup expansion. Their purpose is to show that every finite semigroup is the image of a semigroup in which the right stabilizers are idempotent.

The prefix expansion has been a useful tool for investigating various classes of semigroups and how they relate to each other, in particular semigroups that are not regular. For example in [11] and [12], Hollings classifies all semigroups whose prefix expansions are left ample monoids or left restriction monoids. The prefix expansion has also motivated new expansions. Szendrei observes in [21] that for a group, there is a simpler construction for the prefix expansion than that given by Rhodes. This led to a new expansion, called the Szendrei expansion, which differs from the prefix expansion for arbitrary semigroups. Fountain and Gomes determine when a monoid has a Szendrei expansion that is a left ample monoid or a left restriction monoid [4].

Graph expansions constitute another major branch of semigroup expansion literature and this thesis belongs to this branch. Margolis and Meakin introduced the first graph expansion, that of a group, in [17]. Starting with a group G generated by a set Σ , they use marked pieces of the group’s Cayley graph as the building blocks to form

a new semigroup. We will refer to this construction as the group graph expansion. The results of Margolis and Meakin for group graph expansions are of three different types. First, they describe its structural properties: the group graph expansion is an E -unitary inverse monoid with maximal group image G ; it is Σ -generated as an inverse monoid; its Green's classes can be described using graph isomorphisms; it also has many finiteness properties, including having a finite subgroup structure identical to that of G . Second, they look at the categorical role of the group graph expansion. They establish that it is an initial object in the category of Σ -generated inverse monoids with maximal group image G . Third, they give a succinct inverse monoid presentation for the group graph expansion.

The group graph expansion construction has been generalized to monoids, inverse semigroups, and ordered groupoids. In all of these settings, pieces of the Cayley graph are used as building blocks for elements. However, each setting also has its own unique flavor that influences the graph expansion's structural properties and determines whether nice categorical or presentation properties can be deduced.

In terms of the recipe for constructing elements, the group graph expansion generalizes most easily to monoids. This being said, little attention has actually been given to monoid graph expansions. Instead, stronger results have been obtained by restricting to special types of monoids. Gould considers the monoid graph expansions of right cancellative monoids in [7]. She determines both structural and categorical properties. If S is a Σ -generated right cancellative monoid, then its monoid graph expansion is a proper left ample monoid; it is Σ -generated in the category of proper left ample monoids; its maximal right cancellative monoid image is S . Her categorical result is that the monoid graph expansion of a right cancellative monoid is an initial object in the category of Σ -generated proper left ample monoids.

In a later paper, Gomes and Gould investigate monoid graph expansions of unipotent monoids [6]. Their findings follow a similar pattern to the right cancellative case. They show that if S is a Σ -generated unipotent monoid, then its monoid graph expansion is a proper weakly left ample monoid. Their categorical result is that the

unipotent monoid graph expansion is an initial object in the category of Σ -generated proper weakly left ample monoids.

In general the monoid graph expansion does not have the nice generating set properties that the group graph expansion has: namely, if T is a nontrivial monoid generated by a set Λ , then the monoid graph expansion is not Λ -generated. In [3], Elston introduces a subsemigroup of the monoid graph expansion, which she refers to as the monoid Cayley expansion, which is Λ -generated. However Elston's actual focus is how various semigroup expansions can be described using the language of derived categories. Elston considers the derived category of the homomorphism from a monoid expansion to the original monoid. She shows that the monoid Cayley expansion is the largest monoid expansion in which the local monoids of the derived category are semilattices. She also notes that this construction can be modified for semigroups to yield an expansion that maintains the generating set.

Lawson, Margolis, and Steinberg provide a description of an inverse semigroup graph expansion in [15]. They use subgraphs of Schützenberger graphs as building blocks for elements. Because of the additional properties of the Schützenberger graphs, they are able to obtain categorical and presentation results for the inverse semigroup graph expansion, in addition to structural results. With regards to structure, they show that if S is a Σ -generated inverse semigroup, then its inverse semigroup graph expansion is also a Σ -generated inverse semigroup. As in the group case, the \mathcal{R} - and \mathcal{L} -relations can be characterized using graph isomorphisms. Their categorical result is that the inverse semigroup graph expansion is the initial object in the category of Σ -generated inverse semigroups with idempotent pure maps to S . With respect to presentability, the inverse semigroup graph expansion can be presented succinctly in a way that generalizes the group situation.

Gilbert and Miller construct a graph expansion for Σ -generated ordered groupoids in [5]. Their approach generalizes both the group and inverse semigroup constructions. In order to form elements of the expansion, they introduce a version of the Cayley graph of an ordered groupoid that is in fact an analogue of the Schützenberger graph

of an inverse semigroup. Gilbert and Miller’s work focuses on the structural properties of the ordered groupoid graph expansions. For example, they show that the graph expansion of a Σ -generated ordered groupoid is again a Σ -generated ordered groupoid; they prove that the ordered groupoid graph expansion has the same finite subgroup structure as the original groupoid. They also generalize Margolis and Meakin’s maximal group image result: Gilbert and Miller show that a Σ -generated ordered groupoid and its graph expansion share the same level groupoid. (Level groupoids are to ordered groupoids what maximal group images are to inverse semigroups.)

In light of the previous work on graph expansions for groups, monoids, inverse semigroups, and ordered groupoids, the aim of this thesis is to describe a graph expansion for semigroups. We wish to investigate its properties and show how it relates to the other graph expansions.

We will start in Chapter 2 by providing the necessary terminology and notation. In particular, we introduce the term semigroup system to refer to a semigroup S , a set Σ which generates S , and a map $\Sigma^+ \rightarrow S$. Similarly, we define group and monoid systems. These will be the input for the graph expansion constructions.

In the first section of Chapter 3, we describe the group and monoid graph expansions, since the semigroup graph expansion construction is most closely related to these. We give the basic properties of the group and monoid expansions. Following this, we also include the major findings of Gould for right-cancellative monoids and of Gomes and Gould for unipotent monoids.

Having laid this foundation, we modify the graph expansion construction so that it can be used for semigroup systems. This gives the semigroup graph expansion. As we did for groups and monoids, we describe its basic properties. We establish that the semigroup graph expansion is a functor from the category of Σ -generated semigroups to the category of semigroups with generating sets. We also justify our use of the term “expansion” to describe the graph expansion.

In general, the semigroup graph expansion of a Σ -generated semigroup is not Σ -generated. This differentiates the semigroup graph expansion from most of the

expansions discussed thus far, which have the special property that they are functors from Σ -generated objects to Σ -generated objects. This motivates us to define a new expansion, which we call the semigroup path expansion, that preserves the generating set Σ . We prove that the semigroup path expansion is a subsemigroup of the semigroup graph expansion. Moreover, as the results in subsequent chapters indicate, semigroup path expansions share many properties of the group graph expansions that the semigroup graph expansion lacks.

In Chapter 4 we investigate the properties of semigroup graph expansions for different types of semigroups. These examples are intended to familiarize the reader with the construction, to illuminate some of the differences between the semigroup graph expansion and other graph expansions, and to provide motivation for many of the general results which we will give in Chapter 5. We start by looking at graph expansions of free semigroups. We can obtain some very specific structural results for this case. For example, we give an alternative presentation for the graph expansion of a free monogenic semigroup, i.e. one that is can be generated with a single generator, with single-element generating set. We also show that path expansions of free semigroups are free semigroups. We move on to looking at many different types of semigroups: semigroup systems of groups, left-zero and right-zero semigroups, direct products with one factor that is a left-zero semigroup, in particular rectangular bands, and semilattices. For each of these settings we try to provide a specific description of idempotent and regular elements, to determine whether the subset of idempotents is a subsemigroup or not, and to show local properties, if any. The implications of the direct product case are particularly interesting: it provides an example of distinct semigroups with semigroup systems that have isomorphic semigroup graph expansions. Throughout this section, we comment about how the results can be extended to path expansions.

Bolstered by the examples, we focus in Chapter 5 on properties applying to all semigroup graph expansions. We start by looking at how properties of elements of a semigroup influence the properties of elements of the graph expansion. For example

we show that if a semigroup S contains a periodic element, then we can find periodic elements in any graph expansion of S . This fact enables us to prove one of the major results about semigroup graph expansions: for E -dense semigroups, the semigroup and its graph expansion have the same maximal group image.

Following this, we cover a number of finiteness properties: in Section 5.3 we show that the semigroup graph expansion is always residually finite, in Section 5.4 we give conditions for it to be finitely generated, and in Section 5.5 we prove that a semigroup graph expansion has the same finite subgroup structure as the semigroup being expanded. In the last part of Chapter 5, we focus on mapping properties of the semigroup graph expansion. We look at subsemigroups and show that if T is a subsemigroup of S , we can construct a map from any graph expansion for T to any graph expansion for S .

We devote Chapter 6 to describing Green's \mathcal{R} -, \mathcal{L} -, \mathcal{H} -, \mathcal{D} -, and \mathcal{J} -relations for semigroup graph expansions. For each relation we give necessary and sufficient graphical criteria that determine when elements are related. We also prove properties about the relations including that, for the semigroup graph expansions, $\mathcal{D} = \mathcal{J}$. At the end of the chapter we construct eggbox diagrams for two examples of semigroup graph expansions.

In Chapter 7, we explore the maps between semigroup graph expansions, semigroup path expansions, and other expansions. We start by introducing Birget and Rhodes' right prefix expansion, in particular its cut-down-to-generators version. The main result of this section is that the semigroup path expansion plays the same role with respect to the semigroup prefix expansion as the group graph expansion plays for the group prefix expansion. In Section 7.2, we investigate monoid and semigroup graph expansions by looking at the homomorphisms between them. We show that the relationship between a semigroup and a monoid system determines what types of maps exist between their respective graph expansions.

Chapter 2

Preliminaries

In this chapter, we review the mathematical concepts and terminology that we will need to describe the various graph expansions. We will start by reviewing the nuances between group, monoid, and semigroup systems because these are the main ingredients of the graph expansions. Following upon this, we give a brief overview of semigroup concepts. This is particularly relevant to Chapter 4, where we describe the semigroup graph expansions of particular types of semigroups, and in Chapters 5 and 6, where we give a detailed account of the general structural properties of semigroup graph expansions. We then cover the generalizations of Green's \mathcal{R} -relation to monoids. This is needed to understand Gomes and Gould's monoid graph expansion results. Following this, we describe concepts related to labeled digraphs. In particular, we describe group, monoid, and semigroup Cayley digraphs and review their properties. Finally, we cover terminology from category theory that will be useful for graph expansions.

2.1 Systems

A *semigroup* is a set S with an associative, binary operation. A *monoid* T is a semigroup which contains an identity element, often denoted by 1. For every $x \in T$, $1x = x1 = x$. A *group* G is a monoid in which every element has an inverse; i.e., if

$x \in G$, then there exists some $y \in G$ such that $xy = yx = 1$. An *inverse semigroup* S is a semigroup with the property that for each $x \in S$, there exists a unique element $y \in S$ such that $xyx = x$ and $yxy = y$. Similarly, an *inverse monoid* T is a monoid with the property that for each $x \in S$, there exists a unique element $y \in S$ such that $xyx = x$ and $yxy = y$.

A subset X of a semigroup S is said to *generate* S as a *semigroup* if all elements of S can be expressed as products of elements of X . A subset X of a monoid T is said to *generate* T as a *monoid* if all non-identity elements of T can be expressed as products of elements of X . If X is a subset of a group, we denote by X^{-1} the set of the inverses of the elements in X . We say that X *generates* G as a *group* if every non-identity element of G can be written as a product of elements from $X \cup X^{-1}$. In the same way, if X is a subset of an inverse semigroup S , we say that X *generates* S as an *inverse semigroup* if all elements of S can be expressed as products of elements of $X \cup X^{-1}$. If X is a subset of an inverse monoid T , we say that X *generates* T as an *inverse monoid* if all non-identity elements of T can be expressed as products of elements of $X \cup X^{-1}$.

Let Σ be a non-empty set. A finite string of symbols from Σ is a *word*. The word which contains no symbols is the *empty word* and is denoted by ϵ . We use the notation Σ^+ for the set of all nonempty, finite words and let $\Sigma^* = \Sigma^+ \cup \epsilon$. Under the operation of concatenation, Σ^+ is the free semigroup on Σ and Σ^* is the free monoid on Σ .

It will be useful to treat generating sets as independent from the objects they generate. A *semigroup system* is a triple $sgp(S, \Sigma, f_S)$, where S is a semigroup, Σ a non-empty set, and f_S a semigroup homomorphism from Σ^+ to S , such that Σf_S generates S as a semigroup. When it is clear which semigroup is referred to, we will write f instead of f_S ; similarly, if it is obvious that a semigroup system is intended, we will drop the “*sgp*” prefix and just write (S, Σ, f) .

For any set Ω , we define Ω^{-1} to be a set of formal inverses for Ω . A *group system* is a triple $gp(G, \Omega, f_G)$, where G is a group, Ω is a set, and f_G is a monoid

homomorphism from $(\Omega \cup \Omega^{-1})^*$ to G such that Ωf_G generates G as a group. A monoid system is a triple $mon(T, \Lambda, f_T)$, where T is a monoid, Λ is a set, and f_T is a monoid homomorphism from Λ^* to T such that Λf_T generates T as a monoid.

A semigroup homomorphism is a function $f : S \rightarrow T$ between semigroups S and T satisfying the property $(xf)(yf) = (xy)f$ for all $x, y \in S$. Group and monoid homomorphisms must satisfy the same condition and map the identity in S to the identity in T . We denote the identity homomorphism on semigroup, monoid, or group X by id_X . If there is no confusion, we refer to it as id . In addition to these, we will also need semigroup, group, and monoid system homomorphisms.

Let (S_1, Σ_1, f_1) and (S_2, Σ_2, f_2) be two semigroup systems. A *semigroup system homomorphism* consists of two semigroup homomorphisms, $\alpha : S_1 \rightarrow S_2$ and $\beta : \Sigma_1^+ \rightarrow \Sigma_2^+$, satisfying $f_1 \circ \alpha = \beta \circ f_2$. This relationship is depicted in Figure 2.1 on page 10. We use the notation $\alpha, \beta : (S_1, \Sigma_1, f_1) \rightarrow (S_2, \Sigma_2, f_2)$ to denote a semigroup system homomorphism. If $\Sigma_1 = \Sigma_2$ and the map β is the identity map, we say that α is Σ_1 -*preserving* and often omit β when referring to the semigroup system homomorphism. If α and β are both surjective, then we say that the semigroup system homomorphism α, β is surjective.

$$\begin{array}{ccc}
 \Sigma_1^+ & \xrightarrow{f_1} & S_1 \\
 \beta \downarrow & & \downarrow \alpha \\
 \Sigma_2^+ & \xrightarrow{f_2} & S_2
 \end{array}$$

Figure 2.1: The maps α, β constitute a semigroup system homomorphism.

Given a set X , an *equivalence relation* R on X is a subset of $X \times X$ that satisfies the following properties:

1. $(x, x) \in R$ for all $x \in X$;
2. $(x, y) \in R$ if and only if $(y, x) \in R$;
3. if $(x, y), (y, z) \in R$, then $(x, z) \in R$.

Suppose there is a binary operation on X . We call an equivalence relation that preserves this operation a *congruence relation*; i.e. R is a congruence relation if and only if for all $(x, y), (w, z) \in R$, we have that $(xw, yz) \in R$. Congruences are naturally ordered by inclusion, and we shall use certain congruences that are minimal in the set of congruences that possess some special property. For example, a congruence relation over X is called a *minimal group congruence* if it is the smallest congruence on X such that the congruence classes form a group.

A semigroup system (S, Σ, f_S) induces a congruence relation R_S on Σ^+ defined by $R_S = \{(u, v) \mid uf_S = vf_S\}$. A *semigroup presentation* of S with respect to the generating set Σ is a pair $\langle \Sigma \mid R \rangle$ such that $R \subseteq R_S$ and the minimal congruence relation containing R is R_S . A monoid system (T, Λ, f_T) induces a congruence relation R_T on Λ^* defined by $R_T = \{(u, v) \mid uf_T = vf_T\}$. A *monoid presentation* $\text{mon}\langle \Lambda; R \rangle$ of a monoid T is a pair $\langle \Lambda \mid R \rangle$ such that $R \subseteq R_T$ and the minimal congruence relation induced by R is R_T . The congruence classes of R_T correspond to the elements of T . A group system (G, Ω, f_G) induces a congruence relation R_G on $(\Omega \cup \Omega)^*$ defined by $R_G = \{(u, v) \mid uf_G = vf_G\}$. A *group presentation* $\text{gp}\langle \Omega; R \rangle$ of a group G is a pair $\langle \Omega \mid R \rangle$ such that $R \subseteq R_G$ and the minimal congruence relation containing R is R_G . The congruence classes of R_G correspond to the elements of G .

A subset X of a semigroup S is a *subsemigroup* of S if it forms a semigroup under the operation inherited from S . We use the notation $X \leq S$ to denote a subsemigroup. If the subsemigroup is a monoid, we call it a *submonoid*. If the subsemigroup is a group, we call it a *subgroup*. If S is a semigroup that is also a monoid, we note that its submonoids need not have the same identity as S . A subsemigroup (submonoid, subgroup) X of S is called a *retract* if there is an endomorphism of S that maps surjectively to X and is the identity when restricted to X .

2.2 Semigroup Terminology

Let S be a semigroup and $c, d \in S$. The element c is *regular* if there exists some $x \in S$ such that $cxc = c$. The subset of regular elements is denoted $\text{Reg}(S)$. We say that S is regular if $S = \text{Reg}(S)$. The elements c and d are *inverses* if $cdc = c$ and $dcd = d$. The element c is said to be an *idempotent* if $c^2 = c$. The subset of idempotents is denoted $E(S)$. (Although “ E ” is also used to convey the edge set of a graph, the intended meaning should be clear from the context.) Idempotents can be used to give an alternative characterization of inverse semigroups: a regular semigroup S is inverse if and only if all elements of $E(S)$ commute.

Periodic elements generalize idempotents. The element c is *periodic* (or in some texts *torsion*) if there exists some $m, n \in \mathbb{N}$ such that $c^m = c^{m+n}$. If the numbers m and n are the smallest numbers with this property, we call them the *index* and *period* of the element c . A periodic element is called *aperiodic* if the period is 1.

An element $x \in S$ is *indecomposable* if there exist no $a, b \in S \setminus \{1\}$ such that $x = ab$. As the etymology indicates, *decomposable* elements are those that are not indecomposable. If an element $z \in S$ has the property that for all $c \in S$, we have $zc = c$, then z is a *left identity* of S . If $z \in S$ has the property that for all $c \in S$, $zc = z$, we call z a *left zero* of S . *Right identities* and *right zeroes* are dually defined. An element z is the *identity* of S if it is both a right and left identity; it is the *zero* of S if it is both a right and left zero. The identity and the zero, should they exist, are unique. If S is not a monoid, we use the notation S^1 to show that we have adjoined an identity to S . In the case where S is a monoid to begin with, we will use the convention that $S = S^1$. We say that S is *residually finite* if for every $x, y \in S$, there exists some semigroup T and a map $\alpha : S \rightarrow T$ such that $x\alpha \neq y\alpha$.

A *left-zero semigroup* L is a semigroup comprised entirely of left zeros; similarly, a *right-zero semigroup* R contains only right zeros. A *rectangular band* S is a semigroup that can be written as a direct product $L \times R$, where L and R are left- and right-zero semigroups respectively. A *semilattice* is a commutative semigroup in which every element is idempotent.

A non-empty subset T of a semigroup S is called *unitary* in S if for all elements $c \in S$ and $t \in T$, $ct \in T$ implies $c \in T$ and $tc \in T$ implies $c \in T$. A semigroup S is called *E -unitary* if the set $E(S)$ is unitary. A semigroup S is *E -dense* if for every $x \in S$, there exists some $y \in S$ such that xy is an idempotent. Some authors use the term *E -inverse* instead of *E -dense*.

We say that a semigroup S is *locally inverse* if for every idempotent $e \in E(S)$, the subsemigroup eSe is an inverse semigroup. We can use the adjective local in other contexts as well: given a property P , we say that S is *locally P* if for every idempotent $e \in E(S)$, the subsemigroup eSe has property P .

Green's relations, \mathcal{R} , \mathcal{L} , \mathcal{H} , \mathcal{J} , and \mathcal{D} are equivalence relations on semigroups. They have been well studied and the theory about them is very rich. The most commonly used Green's relations are the right and left relations, \mathcal{R} and \mathcal{L} . If $x, y \in S$, we say that $x\mathcal{R}y$ if $xS^1 = yS^1$. Alternatively, if $x \neq y$, then $x\mathcal{R}y$ if and only if there exist some $a, b \in S$ such that $xa = y$ and $yb = x$. The left relation is defined dually. The notation R_x and L_x stands for the \mathcal{R} - and \mathcal{L} -classes of x . The \mathcal{H} - and \mathcal{D} -relations are defined using the \mathcal{R} - and \mathcal{L} -classes: $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$ and $\mathcal{D} = \mathcal{L}\mathcal{R} = \mathcal{R}\mathcal{L}$. The remaining relation is the \mathcal{J} -relation: $x\mathcal{J}y$ if and only if $S^1xS^1 = S^1yS^1$.

We will also need terminology about semigroups interacting with other algebraic objects. If S is a semigroup and X a set, we say that S *acts on X on the left* if there is a map from $S \times X \rightarrow X$ (denoted by $(a, b) \mapsto a \cdot b$) with the following property: for all $s, t \in S$ and $x \in X$, $(st) \cdot x = s \cdot (t \cdot x)$.

Given a semigroup S , a group G is called a *universal group* of S if there is a homomorphism $\sigma : S \rightarrow G$, such that for every group H and homomorphism $\alpha : S \rightarrow H$, there exists a unique homomorphism β making the diagram below commute.

$$\begin{array}{ccc} S & \xrightarrow{\alpha} & H \\ \sigma \downarrow & & \nearrow \beta \\ G & & \end{array}$$

The group G is unique up to isomorphism. If the homomorphism σ is surjective, then we call such a group the *maximal group image* of S . If it exists, it can be obtained as the quotient of the *minimal group congruence* on S . The following are examples of maximal group images: if a semigroup S is in fact a group, then it is its own maximal group image; if S is an inverse semigroup, then its maximal group image is the group isomorphic to the quotient of S by the congruence relation induced by the set

$$\{(a, b) \mid \text{there exists some } e \in E(S) \text{ such that } ea = eb\}.$$

The following result about E -dense semigroups will be particularly useful:

Lemma 2.2.1. *Let S be an E -dense semigroup. Then S possesses a maximal group image.*

Proof: See Hall and Munn for an existence proof [9]. See Mitsch for an explicit construction of the group using the minimal group congruence [19]. ■

2.3 Monoid Terminology

Gomes and Gould investigate graph expansions of specific types of monoids in [7] and [6]. In this section we wish to supply the terminology necessary to understand their results.

A monoid T is *right cancellative* if for all $a, b, c \in T$, $ac = bc$ implies $a = b$. We say that T is *unipotent* if it contains a single idempotent. Just as a minimal group congruence exists on certain semigroups, there is a minimal right cancellative congruence and a minimal unipotent congruence on certain monoids. All of these congruences are denoted by σ .

When studying the graph expansions of right cancellative and unipotent monoids, Gomes and Gould use generalizations of Green's \mathcal{R} -relation. We describe these relations and the monoids they characterize. The *right star relation*, \mathcal{R}^* , is defined by $c\mathcal{R}^*d$ if and only if for all $x, y \in T$, $xc = yc$ if and only if $xd = yd$. The *right tilde*

relation, $\tilde{\mathcal{R}}$, is defined by $c\tilde{\mathcal{R}}d$ if and only if c and d have the same idempotent left identities, i.e.

$$\{x | xc = c \text{ and } x^2 = x\} = \{y | yd = d \text{ and } y^2 = y\}.$$

We note these relations are contained in each other: $\mathcal{R} \subseteq \mathcal{R}^* \subseteq \tilde{\mathcal{R}}$. We can use the \mathcal{R}^* - and $\tilde{\mathcal{R}}$ -relations to describe different types of monoids. We say that T is *left adequate* if $E(T)$ is a semilattice and every \mathcal{R}^* -class contains an idempotent. In this case, the idempotent of each \mathcal{R}^* -class is unique. For $a \in T$, we denote the idempotent of \mathcal{R}_a^* by a^+ . The monoid T is *left ample* if it is left adequate and for all $a \in T$ and $e \in E(T)$, we have that $ae = (ae)^+a$. Let σ be the minimal right-cancellative congruence on T . We call T *proper left ample* if $\mathcal{R}^* \cap \sigma$ is the identity congruence.

We say that T is *weakly left adequate* if $\tilde{\mathcal{R}}$ is a left congruence, $E(T)$ is a semilattice, and every $\tilde{\mathcal{R}}$ -class contains an idempotent. In this case, there is a unique idempotent in each $\tilde{\mathcal{R}}$ -class: we denote the idempotent of $\tilde{\mathcal{R}}_a$ by a^+ . Furthermore, T is *weakly left ample* if it is weakly left adequate and for all $a \in T$ and $e \in E(T)$, we have that $ae = (ae)^+a$. Let σ be the minimal unipotent congruence on T . We call T *proper weakly left ample* if $\tilde{\mathcal{R}} \cap \sigma$ is the identity congruence.

2.4 Labeled Digraphs, in Particular Cayley Digraphs

A *labeled directed graph* Γ consists of three sets and three maps:

- a non-empty set of vertices, $V(\Gamma)$;
- a set of edges, $E(\Gamma)$;
- a set of edge labels, $\Sigma(\Gamma)$;
- an initial map $\iota : E(\Gamma) \rightarrow V(\Gamma)$;
- a terminal map $\tau : E(\Gamma) \rightarrow V(\Gamma)$;
- a surjective label-assigning map $\lambda : E(\Gamma) \rightarrow \Sigma(\Gamma)$.

We will also not allow a labeled digraph to contain multiple edges with the same initial vertex, same terminal vertex, and same label. If edges e and f are such that $e\iota = f\iota$, $e\tau = f\tau$, and $e\lambda = f\lambda$, then $e = f$. We will use the abbreviation “*digraph*” for directed graph. In general, we will drop the word “labeled”, since all of the digraphs with which we will work are labeled. A digraph is finite if the sets $E(\Gamma)$ and $V(\Gamma)$ are finite.

A digraph P is a *subdigraph* of a digraph Γ , which we denote by $P \subseteq \Gamma$, if $V(P) \subseteq V(\Gamma)$, $E(P) \subseteq E(\Gamma)$, $\Sigma(P) \subseteq \Sigma(\Gamma)$ and the maps for P are the same as those for Γ , when restricted to P . We denote the set of all the subdigraphs of Γ by $\mathcal{P}(\Gamma)$. This set forms a semilattice with the operation union.

We say that a digraph Γ is *deterministic* if for any $c \in V(\Gamma)$ and $r \in \Sigma(\Gamma)$, there is at most one edge e such that $e\iota = c$ and $e\lambda = r$. The digraph Γ is *complete* if for any $c \in V(\Gamma)$ and $r \in \Sigma(\Gamma)$, there is at most one edge e such that $e\iota = c$ and $e\lambda = r$. If a digraph Γ is complete and deterministic, then we can describe an edge e uniquely by the pair $(e\iota, e\lambda)$. In diagrams, we will also use the notation $\underset{e\iota}{\bullet} \xrightarrow{e\lambda} \underset{e\tau}{\bullet}$ to represent the edge e .

If a and b are two vertices of Γ , a *path* from a to b is a finite sequence of edges, e_1, e_2, \dots, e_n such that

$$\underset{a=e_1\iota}{\bullet} \longrightarrow \underset{e_1\tau=e_2\iota}{\bullet} \longrightarrow \dots \longrightarrow \underset{e_{n-1}\tau=e_n\iota}{\bullet} \longrightarrow \underset{e_n\tau=b}{\bullet}.$$

This path is *labeled* by the word $(e_1\lambda)(e_2\lambda)\dots(e_n\lambda)$. We allow a path to pass through the same vertex or edge multiple times. The *empty path* at a vertex a is a path consisting of no edges. A *minimal path* is a path which does not pass through any vertex more than once. We will regard empty paths as minimal paths. A *cycle* is a path which has the same start and end vertex. We will denote a path that starts at vertex a , ends at vertex b , and is labeled by a word w by $a \xrightarrow{w} b$. Given any path, its *underlying digraph* is the minimal subdigraph that contains the path. We will use the notation $[a \xrightarrow{w} b]$ to refer to the underlying digraph of the path $a \xrightarrow{w} b$.

If $a, b \in V(\Gamma)$, we say that b is *accessible* from a if there is an $a \rightarrow b$ path in Γ . A digraph Γ is *rooted* at a if every vertex in Γ is accessible from a , in which case we refer to a as a root of Γ . We say that a digraph Γ is *strongly connected* if each of its vertices is a root. Let $P \subseteq \Gamma$ and $c \in V(P)$. The *subdigraph of P accessible from c* , denoted P_c^\uparrow , is the maximal subdigraph of P that is rooted at c . The *subdigraph of P potentially accessible from c* , denoted $P_c^{\uparrow\uparrow}$, is the intersection of P and Γ_c^\uparrow with any isolated vertices not equal to c removed. We illustrate accessibility and potential accessibility in Figure 2.2.

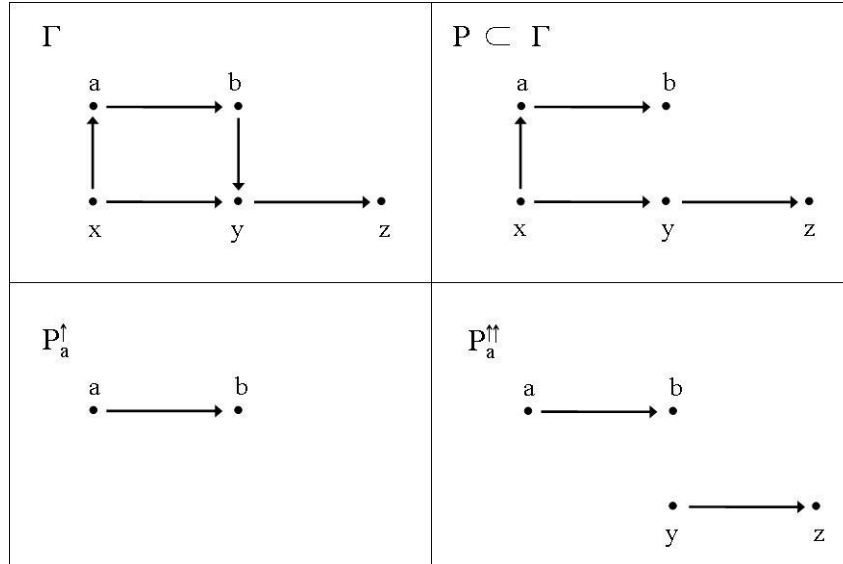


Figure 2.2: The digraph P is a subdigraph of Γ and P contains the vertex a . We show P_a^\uparrow , the digraph accessible from a , and $P_a^{\uparrow\uparrow}$, the digraph potentially accessible from a .

If P and Q are two subdigraphs of a digraph Γ , we say that P and Q are *disjoint* if $V(P) \cap V(Q) = \emptyset$. For the next definition, we will assume that the digraph Γ is deterministic, so that we can describe its edges with the notation (v, r) . We say that the digraph P is *weakly connected* if for all vertices $a, b \in V(P)$ there exist vertices $a = v_1, v_2, \dots, v_m = b \in V(P)$ and edges $e_1, e_2, \dots, e_{m-1} \in E(\Gamma)$ such that for

$1 \leq i \leq m-1$, either $e_i\iota = v_i$ and $e_i\tau = v_{i+1}$ or $e_i\iota = v_{i+1}$ and $e_i\tau = v_i$. A subdigraph $P \subseteq \Gamma$ is called a *component* of Γ if P is a maximal weakly connected subdigraph. If P and Q are distinct components, then they are disjoint.

A digraph is a *labeled graph* if for any edge e there exists an edge $f \in E(\Gamma)$ such that $e\iota = f\tau$; and $e\tau = f\iota$. As we did for digraphs, we will drop the adjective “labeled” for graphs, since all of the graphs with which we will work are labeled. Also for graphs, being weakly connected is equivalent to being strongly connected. Thus we will use the phrase “connected” in this case. A digraph P is a *subgraph* of a digraph Γ if P is both a subdigraph and a graph.

Some of the graphs with which we will work also have the property that we can partition their edge label sets $\Sigma(\Gamma) = \Sigma(\Gamma)^+ \cup \Sigma(\Gamma)^-$ and then find a bijection $\alpha : \Sigma(\Gamma)^+ \rightarrow \Sigma(\Gamma)^-$ such that for any edge e there exists an edge $f \in E(\Gamma)$ with the following properties:

- $e\iota = f\tau$;
- $e\tau = f\iota$;
- if $e\lambda \in \Sigma(\Gamma)^+$, then $e\lambda \circ \alpha = f\lambda$;
- if $e\lambda \in \Sigma(\Gamma)^-$, then $e\lambda \circ \alpha^{-1} = f\lambda$.

Recalling that there is exactly one edge with a given start vertex, terminal vertex, and edge label, we see that for each edge e , there is exactly one f as described above. We say e and f are inverse edges and we can denote f by e^{-1} . Note that $e = (e^{-1})^{-1}$. We call a graph with this property an inverse graph.

Assume Γ is a graph with respect to the partition $\Sigma(\Gamma)^+ \cup \Sigma(\Gamma)^-$ and the bijection $\alpha : \Sigma(\Gamma)^+ \rightarrow \Sigma(\Gamma)^-$. If P is a subdigraph of Γ , the *graph completion* of P , denoted by \overline{P} , is the minimal subgraph of Γ containing P . We can also describe \overline{P} outright; it consists of the sets:

- $V(\overline{P}) = V(P)$;
- $E(\overline{P}) = \{e \mid e \in E(P) \text{ or } e^{-1} \in E(P)\}$;

- $\Sigma(\overline{P}) = \{r \mid r = e\lambda \text{ and } e \in E(\overline{P})\};$

and the maps inherited from Γ . If P is already a subgraph, then $P = \overline{P}$.

A *digraph morphism* from a digraph Γ to a digraph Γ' is a morphism θ comprised of three maps:

$$\theta_V : V(\Gamma) \rightarrow V(\Gamma'), \quad \theta_E : E(\Gamma) \rightarrow E(\Gamma'), \quad \text{and} \quad \theta_\Sigma : \Sigma(\Gamma) \rightarrow \Sigma(\Gamma')$$

with the property that for any $e \in E(\Gamma)$,

$$e\iota \circ \theta_V = e\theta_E \circ \iota', \quad e\tau \circ \theta_V = e\theta_E \circ \tau', \quad e\lambda \circ \theta_\Sigma = e\theta_\Sigma \circ \lambda'.$$

Two digraphs Γ and Γ' are isomorphic if each of the three maps is a bijection. We say that the map θ is label-preserving if the map θ_Σ is the identity map.

We shall also consider left semigroup actions on digraphs. We will assume that the digraph Γ is deterministic, so that we can describe its edges with the notation (v, r) . We say that S *acts on the digraph Γ on the left* if S acts on $V(\Gamma)$ on the left and for each $s \in S$ and $(v, r) \in E(\Gamma)$, we have that $s \cdot ((v, r)\tau) = (s \cdot v, r)\tau$. If $s \in S$ and $P \subseteq \Gamma$, then $s \cdot P$ is the graph comprised of the following sets:

- $V(s \cdot P) = s \cdot V(P);$
- $E(s \cdot P) = \{(s \cdot v, r) \mid (v, r) \in E(P)\};$
- $\Sigma(s \cdot P) = \Sigma(P).$

In this thesis, we are primarily concerned with Cayley digraphs and their subdigraphs. Semigroup, monoid, and group systems can be interpreted graphically using Cayley digraphs. The *Cayley digraph* of a semigroup system (S, Σ, f_S) , denoted $\text{Cay}(S; \Sigma)$, is the digraph comprised of the following:

- Vertex set = S ;
- Edge set = $\{(x, s) \mid x \in S \text{ and } s \in \Sigma\};$
- Edge Label set = Σ ;

- Maps: $(x, s)\iota = x$, $(x, s)\tau = x(sf_S)$, $(x, s)\lambda = s$.

The Cayley digraph of a monoid system is defined similarly. For a group presentation (G, Ω, f_G) , we modify the edge and edge label sets as follows:

- Edge set = $\{(x, s) : x \in G \text{ and } s \in \Omega \cup \Omega^{-1}\}$;
- Edge Label set = $\Omega \cup \Omega^{-1}$;

and extend the function f_G to Ω^{-1} as described in Section 2.1.

Cayley digraphs have a number of useful properties which will be important to us. First, all Cayley digraphs, regardless of the type, are deterministic. In the Cayley digraphs of group systems, there is also exactly one edge of each label entering each vertex. Cayley digraphs of group systems are graphs; Cayley digraphs of monoid or semigroup systems of groups are strongly connected. Cayley digraphs of monoid systems are rooted at 1. Cayley digraphs of semigroups may have disjoint components.

Since the vertices of the Cayley digraph $\text{Cay}(S; \Sigma)$ are the elements of S , there is a left action of S on $V(\text{Cay}(S; \Sigma))$. This induces a left action of S on the edge set, $E(\text{Cay}(S; \Sigma))$ defined by $x \cdot (v, r) \mapsto (xv, r)$. We will refer to these actions as left multiplication or left translation by S and write xc instead of $x \cdot c$, $x(d, r)$ instead of $x \cdot (d, r)$. Moreover, since

$$x \cdot ((d, r)\tau) = x \cdot (d(rf)) = (x \cdot d, r)\tau,$$

we see that this leads to a left semigroup action of S on $\text{Cay}(S; \Sigma)$. If $P \subseteq \text{Cay}(S; \Sigma)$, then $x \cdot P$ denotes the digraph obtained by acting on P by x , i.e. translating all vertices and edges of P by x . We usually write xP instead of $x \cdot P$. We note that both groups and monoids have corresponding left actions for their respective Cayley digraphs.

2.5 Semigroup Expansions

A *category* \mathbf{C} consists of two collections: a collection of objects \mathcal{O} and a collection of arrows \mathcal{A} . Each arrow $f \in \mathcal{A}$ has a unique source object and a unique target object, denoted by $\mathbf{source}(f)$ and $\mathbf{target}(f)$ respectively. There is also a partially defined binary operation on arrows: if $f, g \in \mathcal{A}$ are such that $\mathbf{target}(f) = \mathbf{source}(g)$, then $f \circ g$ is an arrow in \mathcal{A} with $\mathbf{source}(f \circ g) = \mathbf{source}(f)$ and $\mathbf{target}(f \circ g) = \mathbf{target}(g)$. The collections \mathcal{A} and \mathcal{O} must satisfy two axioms:

1. for every object $o \in \mathcal{O}$, there is an arrow $id_o \in \mathcal{A}$ such that if f and g are arrows with $\mathbf{source}(f) = o$ and $\mathbf{target}(g) = o$, then $id_o \circ f = f$ and $g \circ id_o = g$;
2. if $f, g, h \in \mathcal{A}$ are such that $f \circ g$ and $g \circ h$ are defined, then $(f \circ g) \circ h = f \circ (g \circ h)$.

The categories relevant to graph expansions are:

GRP	groups with group homomorphisms;
MON	monoids with monoid homomorphisms;
SGP	semigroups with semigroup homomorphisms;
INV	inverse semigroups with semigroup homomorphisms;
INV-MON	inverse monoids with monoid homomorphisms;
SGP-SYS	semigroup systems and semigroup system homomorphisms;
GRP_Ω	Ω-generated group systems and Ω-preserving group system homomorphisms;
MON_Λ	Λ-generated monoid systems and Λ-preserving monoid system homomorphisms;
SGP_Σ	Σ-generated semigroup systems and Σ-preserving semigroup system homomorphisms;

INV_Σ	Σ-generated inverse semigroup systems and Σ-preserving inverse semigroup system homomorphisms.
INV-MON_Λ	Σ-generated inverse monoids and Σ-preserving inverse monoid system homomorphisms.

If \mathbf{B} and \mathbf{C} are categories, a *functor* $F : \mathbf{B} \rightarrow \mathbf{C}$ is a map between \mathbf{B} and \mathbf{C} that satisfies the following: F assigns each object in \mathbf{B} to an object in \mathbf{C} and each arrow in \mathbf{B} to an arrow in \mathbf{C} such that $F(id_o) = id_{F(o)}$ and $F(f \circ g) = F(f) \circ F(g)$.

Suppose we have two functors: $F : \mathbf{B} \rightarrow \mathbf{C}$ and $G : \mathbf{B} \rightarrow \mathbf{C}$. A *natural transformation* $\tau : F \rightarrow G$ is a function which assigns to each object $o \in B$ an arrow $\tau_o : F(o) \rightarrow G(o)$ of \mathbf{C} which makes the diagram shown in Figure 2.3 commute (in the diagram, we assume that f is any arrow of \mathbf{B} , that $\text{source}(f) = x$ and $\text{target}(f) = y$):

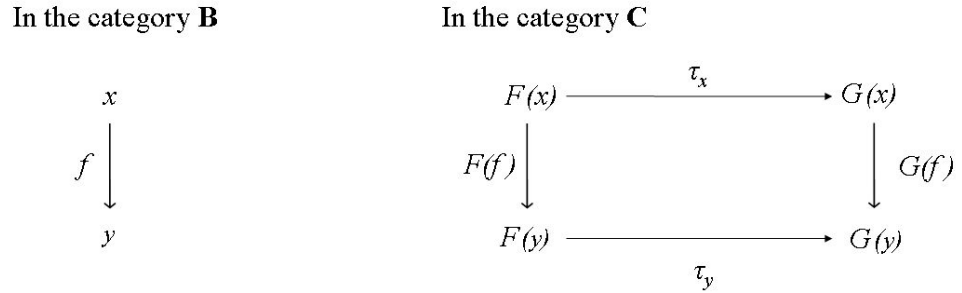


Figure 2.3: A natural transformation between the functors $F, G : \mathbf{B} \rightarrow \mathbf{C}$

We say that a functor F from **SGP**, **SGP-SYS**, or one of their subcategories (denote the domain by \mathbf{A}), to **SGP**, **SGP-SYS**, or one of their subcategories (denote the codomain by \mathbf{B}) is a *semigroup expansion* if F satisfies these two properties:

1. there is a natural transformation $\epsilon : F \rightarrow id$ where id is the identity functor on the objects of \mathbf{A} ;
2. if $S \in \mathbf{A}$, then the map $\epsilon_S : F(S) \rightarrow S$ is surjective.

Chapter 3

Graph Expansions

In this chapter, we will describe the group, monoid, and semigroup, graph expansions. This description serves two purposes: first, it is insightful to see how the semigroup construction evolves from the monoid construction, which in turn evolved from the group construction. Second, having a clear descriptions of these graph expansions will allow us in Section 7 to show how they are related as subsemigroups and/or images of each other.

We will give the basic structural properties of each graph expansion: what type of algebraic object it is, if it is E -unitary, a description of its inverses, idempotents, and/or regular elements, information about how it is generated, any special congruence properties, and each graph expansion's domain and codomain as a functor. We note that we have left out some categorical and presentation related results for the group graph expansion, as well as for the special cases of the monoid graph expansion. This is because there are no nice analogs for the semigroup graph expansion.

At the end of the chapter, we introduce the semigroup and monoid path expansions. These are subsemigroups of the respective graph expansions. In addition to possessing most of the properties of the graph expansions, they are “nicely” generated in same way that the group graph expansion is. For this reason, we will often be able to draw stronger comparisons between the path expansions and the group graph expansion in later chapters.

3.1 The Group and Monoid Graph Expansions

Margolis and Meakin define the *graph expansion of a group system* (G, Ω, f) , denoted $M_{gp}(G; \Omega)$, as a set of pairs:

$$\{(P, c) \mid P \text{ is a finite, connected subgraph of } \text{Cay}(G; \Omega), \text{ and } 1, c \in V(P)\}$$

with multiplication defined as: $(P, c)(Q, d) = (P \cup cQ, cd)$. Note that the subgraph $P \cup cQ$ is finite, connected, and contains both the vertices 1 and cd , which ensures that the set is closed under the operation. The following properties of the graph expansion of a group are due to Margolis and Meakin [17]:

Theorem 3.1.1. *Let (G, Ω, f) be a group system. Then $M_{gp}(G; \Omega)$*

- (a) *is a monoid with identity element $(\bullet_1, 1)$;*
- (b) *is an inverse monoid; the inverse of (P, c) is $(c^{-1}P, c^{-1})$;*
- (c) *is E-unitary; an element (P, c) is idempotent if and only if $c = 1$;*
- (d) *is generated as an inverse monoid by the set $\{(\bullet_1 \xrightarrow[gf]{g} \bullet, gf) \mid g \in \Omega\}$;*
- (e) *has minimal group congruence $(P, c) \sim (Q, d)$ if and only if $c = d$ and thus has maximal group image G ;*
- (f) *$M_{gp}(_; \Omega)$ is a functor from the category \mathbf{GRP}_Ω to the category $\mathbf{INV-MON}_\Omega$.*

The minimal group congruence on $M_{gp}(G; \Omega)$ determines a homomorphism from $M_{gp}(G; \Omega)$ to G . We denote this homomorphism by σ_G .

The construction for groups easily generalizes to monoids. The *graph expansion of a monoid system* (T, Λ, f) , denoted $M_{mon}(T; \Lambda)$, is the set:

$$\{(P, c) \mid P \text{ is a finite subdigraph of } \text{Cay}(T; \Lambda), P \text{ is rooted at } 1, \text{ and } c \in V(P)\}$$

with the multiplication of elements defined in the same manner as for the graph expansion of groups. In contrast to the group situation, we can say less about the general monoid case. The monoid analog to Theorem 3.1.1 is:

Theorem 3.1.2. *Let (T, Λ, f) be a monoid system. Then $M_{\text{mon}}(T; \Lambda)$ is a monoid with identity element $(\bullet_1, 1)$.*

Gould was able to develop further results by restricting to graph expansions of right cancellative monoids (see [7]). She obtained the following:

Theorem 3.1.3. *Let (T, Λ, f) be a monoid system of a right cancellative monoid T . Then $M_{\text{mon}}(T; \Lambda)$*

- (a) is a proper left ample monoid with identity element $(\bullet_1, 1)$;*
- (b) is E -unitary; an element (P, g) is idempotent if and only if $g = 1$;*
- (c) is generated as a proper left ample monoid by the set $\{(\bullet_1 \xrightarrow[g_f]{g} \bullet, gf) \mid g \in \Lambda\}$;*
- (d) has maximal right cancellative monoid image T .*

Gomes and Gould realized that a similar generalization applies to unipotent monoids (see [6]).

Theorem 3.1.4. *Let (T, Λ, f) be a monoid system of a unipotent monoid T . Then $M_{\text{mon}}(T; \Lambda)$*

- (a) is a proper weakly left ample monoid with identity element $(\bullet_1, 1)$;*
- (b) is E -unitary; an element (P, g) is idempotent if and only if $g = 1$;*
- (c) is generated as a proper weakly left ample monoid by the set*

$$\{(\bullet_1 \xrightarrow[g_f]{g} \bullet, g) \mid g \in \Lambda\};$$
(d) has maximal unipotent monoid image T .

Additionally, they were able to show the converse of Theorem 3.1.4(a):

Theorem 3.1.5. *Let (T, Λ, f) be a monoid system. If $M_{\text{mon}}(T; \Lambda)$ is weakly left ample, then T is unipotent.*

3.2 The Semigroup Graph Expansion

Since Cayley digraphs of semigroups are constructed in the same manner as Cayley graphs of groups and monoids, it is natural to consider generalizing these graph expansions to semigroups. However, this involves defining elements and a multiplication rule without relying on an identity.

As we have seen, when generalizing from groups to monoids, graphs are replaced by 1-rooted digraphs. To move from monoids to semigroups, we will replace 1-rooted digraphs with other rooted digraphs. However, simply requiring “rootedness” is not enough. To see this, suppose P and Q are the subdigraphs $\bullet_c, \bullet_d \subseteq \text{Cay}(S; \Sigma)$ and that $c \neq cd$. The digraphs P and Q are trivially rooted, but $P \cup cQ$ is not rooted, since there is no edge between \bullet_c and \bullet_d . In order to ensure that products are rooted, we will need to modify the operation. We introduce some new notation: if $P \subseteq \text{Cay}(S; \Sigma)$ is a finite subdigraph and $r \in \Sigma$, we denote by P_r^1 the subdigraph of $\text{Cay}(S^1; \Sigma)$ that is the union of P and the edge $(1, r)$. If S is already a monoid and $(1, r) \in E(P)$, then $P = P_r^1$.

To illustrate this notation, consider the Cayley digraph of a three-element right-zero semigroup system $(S, \{r, s, t\}, id)$. This Cayley digraph is shown in Figure 3.1. In Figure 3.2, we show two subdigraphs, P which is rf -rooted and Q which is sf -rooted. Then we show the digraphs P_r^1 , and Q_s^1 .

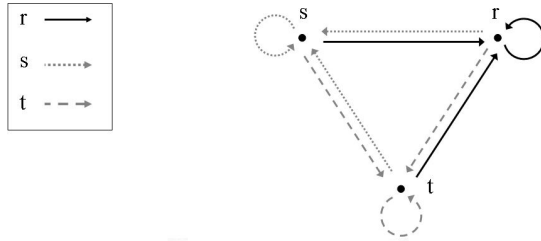


Figure 3.1: The Cayley digraph of the right-zero semigroup system $\{S, \{r, s, t\}, id\}$.

If S is not a monoid, then $P_r^1 \not\subseteq \text{Cay}(S; \Sigma)$. However for any $c \in S$, the translation

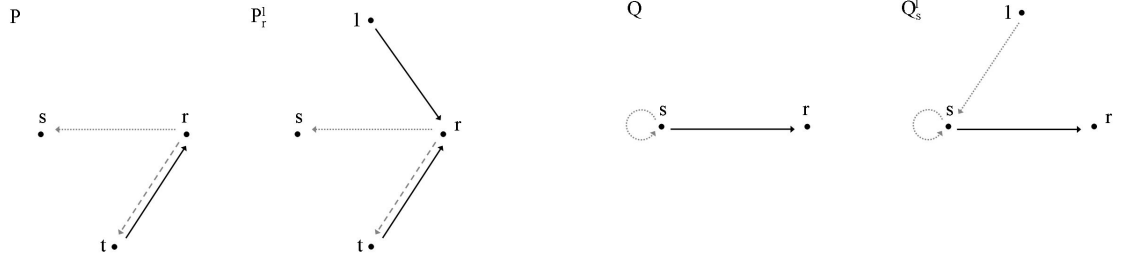


Figure 3.2: The digraphs P_r^1 and Q_s^1 are constructed from P and Q by adding the edges $(1, r)$ and $(1, s)$ respectively. Note that P is r -rooted and Q is s -rooted.

$c(P_r^1) = cP \cup \{(c, r)\}$ lies within $\text{Cay}(S; \Sigma)$. In the case when S is a monoid, we have that $P_r^1 \subseteq \text{Cay}(S; \Sigma)$.

Using this notation, we now define the *graph expansion of a semigroup system* (S, Σ, f) , denoted $\mathcal{M}(S; \Sigma)$, to be the set of triples:

$$\{(r, P, c) \mid r \in \Sigma, P \subseteq \text{Cay}(S; \Sigma) \text{ is finite and rooted at } rf, \text{ and } c \in V(P)\}$$

with multiplication defined by: $(r, P, c)(s, Q, d) = (r, P \cup cQ_s^1, cd)$. This operation can be visualized as shown in Figure 3.3.

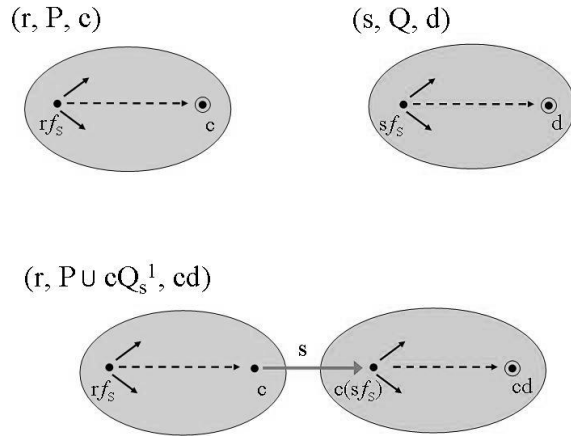


Figure 3.3: Multiplication in the graph expansion of semigroups.

Before describing some basic properties of the semigroup graph expansion, we wish to give a few example of the operation. A very simple case is the system

$(S; \{x, y\}, id)$, where S is a free semigroup on two generators. The Cayley digraph $\text{Cay}(S; \{x, y\})$ is shown in Figure 3.4. We illustrate various products in $\mathcal{M}(S; \Sigma)$ in Figures 3.5 and 3.6. We note that in these figures, as well as in some later ones, we indicate the third entry (i.e., the chosen vertex) not only by stating it (as the third entry), but also by circling the corresponding vertex in the digraph (i.e., the second entry). The information is redundant, but we include it anyway since it makes locating the chosen vertex in the digraph faster.

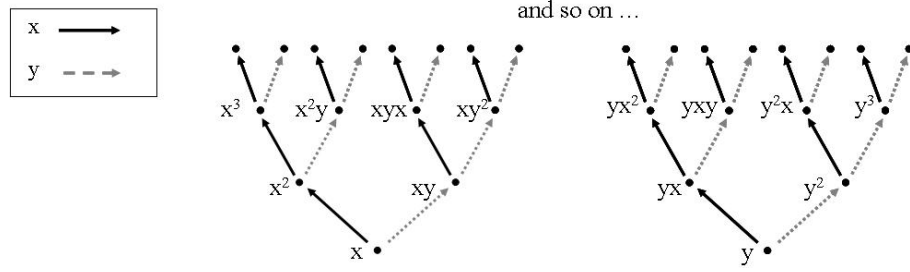


Figure 3.4: The Cayley digraph $\text{Cay}(S; \{x, y\})$ of the free semigroup S which is generated by x and y .

Figure 3.5 illustrates how an edge is added (namely the edge (x^2y, y)) when forming the product. The presence of this edge is necessary for the digraph to be rooted.

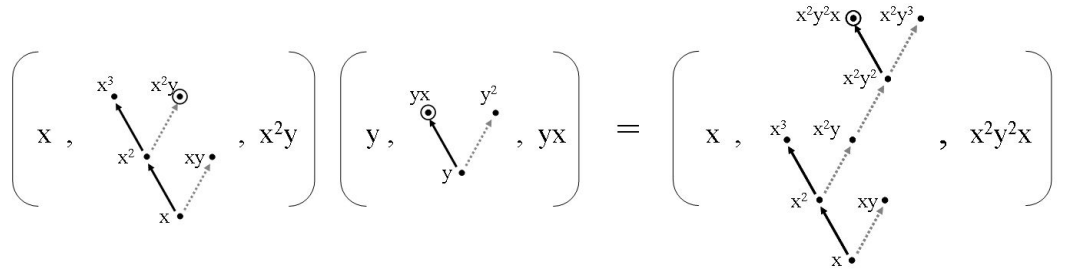


Figure 3.5: An example in which an edge is added to form the product in $\mathcal{M}(S; \{x, y\})$.

In Figure 3.6, we show a product in which no edge is added, since it is already present in the digraph. The edge that did not need to be added is (x^2, y) . In fact, since the digraph in the left factor contains the translation of the digraph from the right factor, we keep the digraph from the left factor in the product. The chosen vertex is the only thing that changes.

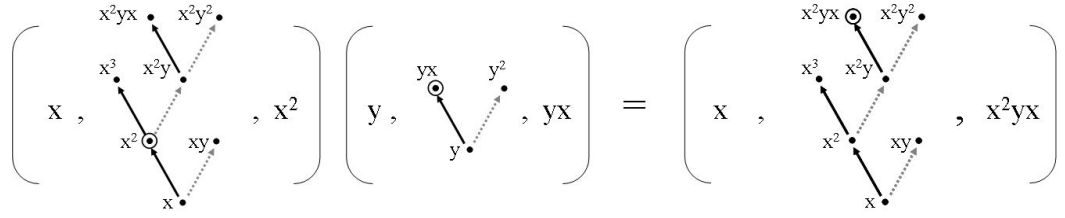


Figure 3.6: (An example where no edge is added when forming the product in $\mathcal{M}(S; \{x, y\})$).

We can see in the above examples how the multiplication rule affects each coordinate of the triple differently. The operation on the first coordinate is left-zero multiplication. The operation on the second coordinate is a modified version of set union. The operation on the third coordinate corresponds to multiplication in S .

We now note basic properties of $\mathcal{M}(S; \Sigma)$:

Theorem 3.2.1. *Let (S, Σ, f) be a semigroup system and let $(r, P, c), (s, Q, d) \in \mathcal{M}(S; \Sigma)$. Then:*

- (a) $\mathcal{M}(S; \Sigma)$ is a semigroup;
- (b) (r, P, c) is idempotent if and only if $c^2 = c$ and $cP_r^1 \subseteq P$;
- (c) (r, P, c) is regular if and only if there exists some $x \in V(P)$ for which there is a non-empty path $c \longrightarrow x$ in P , $xc = c$, and $xP_r^1 \subseteq P$;
- (d) if $(r, P, c)(s, Q, d) = (s, Q, d)(r, P, c)$, then $r = s$;
- (e) the map $\epsilon_S : \mathcal{M}(S; \Sigma) \rightarrow S$ defined by $(r, P, c) \mapsto c$ is a surjective semigroup homomorphism;

Proof: For part (a), observe that $(r, P, c)(s, Q, d) = (r, P \cup cQ_s^1, cd)$. Clearly $P \cup cQ_s^1$ is finite since both P and Q are. The subdigraph $P \cup cQ_s^1$ is rooted at rf since P is rooted at rf , $c \in V(P)$, and cQ_s^1 is rooted at c . Since $d \in V(Q)$, we have that $cd \in V(cQ)$ and it follows that $cd \in V(P \cup cQ_s^1)$. Hence $(r, P, c)(s, Q, d) \in \mathcal{M}(S; \Sigma)$. Showing that the operation is associative is a simple calculation and we omit it.

As the proof of (b) is similar, if slightly simpler, than the proof of (c), we proceed immediately to (c). Suppose (r, P, c) is regular. Then there exists some element (s, Q, d) such that

$$(r, P, c) = (r, P, c)(s, Q, d)(r, P, c) = (r, P \cup cQ_s^1 \cup cdP_r^1, cdc). \quad (3.2.1)$$

Let $x = cd$. From Equation 3.2.1 we see that $(cd)c = c$ and $(cd)P_r^1 \subseteq P$. Finally, since the subdigraph Q_s^1 contains a $1 \rightarrow d$ path, the translated digraph $cQ_s^1 \subseteq P$ contains a $c \rightarrow cd$ path.

Conversely, suppose there is some $x \in V(P)$, such that x is accessible from c in P , $xc = c$, and $xP_r^1 \subseteq P$. There is a word $w \in \Sigma^+$ which labels a $c \rightarrow x$ path in P . Write w as $w = sv$ where $s \in \Sigma$ and $v \in \Sigma^*$. Let Q be the digraph $[sf \xrightarrow{v} wf]$. Note that $wf \in V(Q)$, $c(wf) = x$, and $cQ_s^1 \subseteq P$. These facts imply the following:

$$\begin{aligned} (r, P, c)(s, Q, wf)(r, P, c) &= (r, P \cup cQ_s^1 \cup c(wf)P_r^1, c(wf)c) \\ &= (r, P \cup xP_r^1, xc) \\ &= (r, P, c). \end{aligned}$$

Easy calculations show that (r, P, c) and $(s, Q, wf)(r, P, c)(s, Q, wf)$ are inverses.

We now proceed to part (d). If (r, P, c) and (s, Q, d) commute, we see immediately

that $r = s$:

$$\begin{aligned}
(r, P \cup cQ_s^1, cd) &= (r, P, c)(s, Q, d) \\
&= (s, Q, d)(r, P, c) \\
&= (s, Q \cup dP_r^1, dc).
\end{aligned}$$

Turning to part (e), observe that

$$\begin{aligned}
((r, P, c)(s, Q, d))\epsilon_S &= ((r, P \cup cQ_s^1, cd))\epsilon_S \\
&= cd \\
&= (r, P, c)\epsilon_S(s, Q, d)\epsilon_S.
\end{aligned}$$

To see that the map ϵ_S is surjective, let $c \in S$. There exists some word $w \in \Sigma^+$ such that $wf = c$. Rewrite w as $w = rv$ where $r \in \Sigma$ and $v \in \Sigma^*$. Then $(r, \lfloor rf \xrightarrow{v} c \rfloor, c) \in \mathcal{M}(S; \Sigma)$ and $(r, \lfloor rf \xrightarrow{v} c \rfloor, c)\epsilon_S = c$. ■

Our next goal in this section is to show that the semigroup graph expansion is a functor from \mathbf{SGP}_Σ to $\mathbf{SGP-SYS}$ and that this construction merits the title “expansion”. The first thing we must do is construct a semigroup system for the semigroup graph expansion. In order to do so, we identify its decomposable and indecomposable elements

Lemma 3.2.2. *Let (S, Σ, f) be a semigroup system and let $(r, P, c) \in \mathcal{M}(S; \Sigma)$. Then:*

- (a) *(r, P, c) is decomposable if and only if P contains an edge that terminates at c ;*
- (b) *(r, P, c) is indecomposable if and only if $rf = c$ and there is no cycle in P passing through rf .*

Proof: Looking at (a), suppose (r, P, c) is decomposable. If $c \neq rf$, then clearly there is an edge in P ending in c . Thus consider the case $c = rf$. There exist elements

(r, A, x) and (s, B, y) such that

$$(r, P, rf) = (r, A, x)(s, B, y) = (r, A \cup xB_s^1, xy).$$

If $rf \neq x$, then rf lies on a cycle which passes through x . If $rf = x$, then there is a cycle at rf . In both cases there is an edge entering rf .

For the converse, suppose there is an edge entering c . Thus, there exists some $d \in V(P)$ and $s \in \Sigma$, such that $d(sf) = c$ and $(d, s) \in E(P)$. Since $(r, P, d)(s, \bullet_{sf}, sf) = (r, P, c)$, we have that (r, P, c) is decomposable. Part (b) follows immediately from part (a). ■

We denote the set of indecomposable elements of $\mathcal{M}(S; \Sigma)$ by $\text{Ind}_{\mathcal{M}(S; \Sigma)}$. If the it is clear which graph expansion is referred to, we will abbreviate this notation to $\text{Ind}_{\mathcal{M}}$. Similarly, we will often abbreviate the identity function $\text{id}_{\mathcal{M}(S; \Sigma)}$ as $\text{id}_{\mathcal{M}}$. Since a semigroup is generated by its indecomposable elements, we obtain the following:

Proposition 3.2.3. *Let (S, Σ, f) be a semigroup system. Then $(\mathcal{M}(S; \Sigma), \text{Ind}_{\mathcal{M}}, \text{id}_{\mathcal{M}})$ is a semigroup system.*

Next, we describe a semigroup system homomorphism from $(\mathcal{M}(S; \Sigma), \text{Ind}_{\mathcal{M}}, \text{id}_{\mathcal{M}})$ to (S, Σ, f) . In order to do so, we need a function between the respective generating sets. Using the fact from Lemma 3.2.2(b) that all elements of $\text{Ind}_{\mathcal{M}}$ have the form (r, P, rf) , we define the function $\epsilon_{\Sigma} : \text{Ind}_{\mathcal{M}}^+ \rightarrow \Sigma^+$ to be that induced by the map $(r, P, rf) \mapsto r$.

Proposition 3.2.4. *Let (S, Σ, f) be a semigroup system. Then $\epsilon_S, \epsilon_{\Sigma}$ is a surjective semigroup system homomorphism from $(\mathcal{M}(S; \Sigma), \text{Ind}_{\mathcal{M}}, \text{id}_{\mathcal{M}})$ to (S, Σ, f) .*

Proof: We wish to show that $\text{id}_{\mathcal{M}} \circ \epsilon_S = \epsilon_{\Sigma} \circ f$. Thus, consider an element

$(r_1, P_1, r_1 f)(r_2, P_2, r_2 f) \dots (r_n, P_n, r_n f) \in \text{Ind}_{\mathcal{M}}^+$. Then:

$$\begin{aligned}
& (r_1, P_1, r_1 f)(r_2, P_2, r_2 f) \dots (r_n, P_n, r_n f)(id_{\mathcal{M}} \circ \epsilon_S) \\
&= (r_1, P_1 \cup (r_1 f)P_2 \dots ((r_1 r_2 \dots r_{n-1})f)P_n, (r_1 r_2 \dots r_n)f) \epsilon_S \\
&= (r_1 r_2 \dots r_n)f \\
&= ((r_1, P_1, r_1 f) \epsilon_{\Sigma} (r_2, P_2, r_2 f) \epsilon_{\Sigma} \dots (r_n, P_n, r_n f) \epsilon_{\Sigma}) f \\
&= (r_1, P_1, r_1 f)(r_2, P_2, r_2 f) \dots (r_n, P_n, r_n f)(\epsilon_{\Sigma} \circ f).
\end{aligned}$$

Since both ϵ_S and ϵ_{Σ} are surjective, the pair $\epsilon_S, \epsilon_{\Sigma}$ is a surjective semigroup system homomorphism. ■

In order to show that the semigroup graph expansion construction is a functor, we must establish that it sends a semigroup system homomorphism such as $\varphi, id_{\Sigma} : (S, \Sigma, f_S) \rightarrow (Y, \Sigma, f_Y)$ to a semigroup system homomorphism between the respective graph expansions. We start by showing that the homomorphism φ, id_{Σ} induces a map between Cayley digraphs:

$$\begin{aligned}
\widehat{\varphi} : \text{Cay}(S; \Sigma) &\rightarrow \text{Cay}(Y; \Sigma) & \text{Vertices: } \varphi : S &\rightarrow Y; \\
& & \text{Edges: } (x, r) &\text{to } (x\varphi, r).
\end{aligned}$$

Recall that $\mathcal{P}(\text{Cay}(S; \Sigma))$ is the set of subdigraphs of the Cayley graph $\text{Cay}(S; \Sigma)$. Under the operation of union, it constitutes a semigroup. By restricting to subdigraphs, the map $\widehat{\varphi}$ above induces a map between $\mathcal{P}(\text{Cay}(S; \Sigma))$ and $\mathcal{P}(\text{Cay}(Y; \Sigma))$. We denote this second map by $\widehat{\varphi}$ as well. Using $\widehat{\varphi}$, we obtain a map between graph expansions:

$$\check{\varphi} : \mathcal{M}(S; \Sigma) \rightarrow \mathcal{M}(Y; \Sigma) \quad (r, P, c) \mapsto (r, P\widehat{\varphi}, c\varphi).$$

We also need a map between the generating sets of $\mathcal{M}(S; \Sigma)$ and $\mathcal{M}(Y; \Sigma)$. Recall that this is a map from the set $(\text{Ind}_{\mathcal{M}(S; \Sigma)})^+$ to the set $(\text{Ind}_{\mathcal{M}(Y; \Sigma)})^+$. For each element

$(s, Q, d) \in \mathcal{M}(Y; \Sigma)$, fix a factoring $(s, Q, d) = (s_1, Q_1, d_1) \dots (s_n, Q_n, d_n)$ where each $(s_i, Q_i, d_i) \in \text{Ind}_{\mathcal{M}(Y; \Sigma)}$. We define the map $\check{\beta}_S : (\text{Ind}_{\mathcal{M}(S; \Sigma)})^+ \rightarrow (\text{Ind}_{\mathcal{M}(Y; \Sigma)})^+$ to be the map induced as follows: let $(r, P, c) \in \text{Ind}_{\mathcal{M}(S; \Sigma)}$. Then $\check{\beta}_S$ maps (r, P, c) to the chosen factoring of $(r, P\check{\varphi}, c\varphi)$, i.e. to

$$(r_1, (P\check{\varphi})_1, (c\varphi)_1) \dots (r_n, (P\check{\varphi})_n, (c\varphi)_n).$$

In the following lemma, we show that $\widehat{\varphi}$, $\check{\varphi}$, and $\check{\beta}_S$ have the desired homomorphism properties.

Lemma 3.2.5. *Let (S, Σ, f_S) and (Y, Σ, f_Y) be semigroup systems and let $\varphi : S \rightarrow Y$ be a Σ -preserving semigroup system homomorphism. Then*

- (a) $\widehat{\varphi} : \mathcal{P}(\text{Cay}(S; \Sigma)) \rightarrow \mathcal{P}(\text{Cay}(Y; \Sigma))$ is a semigroup homomorphism;
- (b) $\widehat{\varphi}$ preserves the left action of S on $\text{Cay}(S; \Sigma)$ (i.e., if $c \in S$, and $A \in \mathcal{P}(\text{Cay}(S; \Sigma))$, then $(cA)\widehat{\varphi} = (c\varphi)(A\widehat{\varphi})$.)
- (c) $\check{\varphi} : \mathcal{M}(S; \Sigma) \rightarrow \mathcal{M}(Y; \Sigma)$ is a surjective semigroup homomorphism;
- (d) $\check{\varphi}, \check{\beta}_S : (\mathcal{M}(S; \Sigma), \text{Ind}_{\mathcal{M}(S; \Sigma)}, \text{id}_{\mathcal{M}(S; \Sigma)}) \rightarrow (\mathcal{M}(Y; \Sigma), \text{Ind}_{\mathcal{M}(Y; \Sigma)}, \text{id}_{\mathcal{M}(Y; \Sigma)})$ is a surjective semigroup system homomorphism.

Proof: We start by proving (a). Let $P, Q \in \mathcal{P}(\text{Cay}(S; \Sigma))$. We wish to show that $(P \cup Q)\widehat{\varphi} = P\widehat{\varphi} \cup Q\widehat{\varphi}$. First we consider vertices. Let $x \in V(P\widehat{\varphi} \cup Q\widehat{\varphi})$. There exists some y_x in $V(P)$ or in $V(Q)$ such that $\{\bullet_{y_x}\}\widehat{\varphi} = \{\bullet_x\}$. It follows that $y_x \in V(P \cup Q)$ and hence $x \in V((P \cup Q)\widehat{\varphi})$. Thus $V(P\widehat{\varphi} \cup Q\widehat{\varphi}) \subseteq V((P \cup Q)\widehat{\varphi})$. These steps can be reversed to show $V((P \cup Q)\widehat{\varphi}) \subseteq V(P\widehat{\varphi} \cup Q\widehat{\varphi})$. We conclude that $V(P\widehat{\varphi} \cup Q\widehat{\varphi}) = V((P \cup Q)\widehat{\varphi})$. Since Cayley digraphs are deterministic and $\widehat{\varphi}$ is label-preserving, a similar argument shows that the respective edge sets are equal. Thus $\widehat{\varphi} : \mathcal{P}(\text{Cay}(S; \Sigma)) \rightarrow \mathcal{P}(\text{Cay}(Y; \Sigma))$ is a homomorphism.

Proceeding to (b), we want to show $(cA)\widehat{\varphi} = (c\varphi)(A\widehat{\varphi})$. Again, we will look at vertex and edge sets. Let $x \in V((cA)\widehat{\varphi})$. There exists some $y_x \in V(A)$ such that $(c\{\bullet_{y_x}\})\widehat{\varphi} = \{\bullet_x\}$. Since the function $\widehat{\varphi}$, when restricted to vertices, corresponds with

φ and φ is a homomorphism, we have that:

$$\begin{aligned}
(c\{\bullet\}_{y_x})\widehat{\varphi} &= \{\bullet\}_{cy_x}\widehat{\varphi} \\
&= \{\bullet\}_{(cy_x)\varphi} \\
&= \{\bullet\}_{(c\varphi)(y_x\varphi)} \\
&= (c\varphi)\{\bullet\}_{y_x\varphi} \\
&= (c\varphi)(\{\bullet\}_{y_x}\widehat{\varphi}) \\
&\in V((c\varphi)(A\widehat{\varphi})).
\end{aligned}$$

Thus $V((cA)\widehat{\varphi}) \subseteq V((c\varphi)(A\widehat{\varphi}))$. The reverse inclusion can be obtained by reversing the order of steps, which then gives $V((cA)\widehat{\varphi}) = V((c\varphi)(A\widehat{\varphi}))$. Again, the determinism of the Cayley graph and the fact that $\widehat{\varphi}$ is label-preserving ensure that edge sets are equal. Thus $(cA)\widehat{\varphi} = (c\varphi)(A\widehat{\varphi})$.

We are ready to show (c). Using the results from parts (a) and (b),

$$\begin{aligned}
((r, P, c)(s, Q, d))\check{\varphi} &= ((r, P \cup cQ_s^1, cd))\check{\varphi} \\
&= (r, (P \cup cQ_s^1)\widehat{\varphi}, (cd)\varphi) \\
&= (r, (P \cup cQ \cup \{(c, s)\})\widehat{\varphi}, (cd)\varphi) \\
&= (r, P\widehat{\varphi} \cup (c\varphi)(Q\widehat{\varphi}) \cup \{(c, s)\}\widehat{\varphi}, (c\varphi)(d\varphi)) \\
&= (r, P\widehat{\varphi} \cup (c\varphi)(Q\widehat{\varphi}) \cup \{(c\varphi, s)\}, (c\varphi)(d\varphi)) \\
&= (r, P\widehat{\varphi}, c\varphi)(s, Q\widehat{\varphi}, d\varphi) \\
&= (r, P, c)\check{\varphi}(s, Q, d)\check{\varphi}.
\end{aligned}$$

Thus, $\check{\varphi} : \mathcal{M}(S; \Sigma) \rightarrow \mathcal{M}(Y; \Sigma)$ is a semigroup homomorphism.

To see that it is surjective, let $(r, P, c) \in \mathcal{M}(Y; \Sigma)$. If P contains no edges, then

$rf_y = c$ and $P = \{\bullet\}_c$, whereupon we have that

$$\begin{aligned} (r, \bullet_{rf_S}, rf_S)\check{\varphi} &= (r, \bullet_{(rf_S)\varphi}, (rf_S)\varphi) \\ &= (r, \bullet_{rf_Y}, rf_Y) \\ &= (r, P, c). \end{aligned}$$

If P contains edges, then it contains no isolated vertices. For each edge $(x_i, s) \in E(P)$, we choose a word $w_i \in \Sigma^*$ such that $(rw_i)f_Y = x_i$ and the path $rf_Y \xrightarrow{w_i} x_i$ lies within P . Thus P is the underlying digraph of the following:

$$P = \lfloor \bigcup_{(x_i, s) \in E(P)} rf_Y \xrightarrow{w_i s} x_i(sf_Y) \rfloor.$$

Let $P' \subseteq \text{Cay}(S; \Sigma)$ be the digraph:

$$P' = \lfloor \bigcup_{(x_i, s) \in E(P)} rf_S \xrightarrow{w_i s} (rw_i s)f_S \rfloor.$$

From its construction, $P'\hat{\varphi} = P$. We now wish to show that there is a preimage of c (under φ) in P' . If $c = rf_Y$, then $rf_S \in V(P')$ and $(rf_S)\varphi = rf_Y = c$. If $c \neq rf_Y$, then there is an edge $(x_i, s) \in E(P)$ such that (x_i, s) terminates in c . The path $rf_Y \xrightarrow{w_i s} x_i(sf_Y)$ also terminates in c . Let c' be the vertex $(rw_i s)f_S$. We see that $(rw_i s)f_S \circ \varphi = (rw_i s)f_Y = c$. Thus, $(r, P', c')\check{\varphi} = (r, P, c)$. We conclude that $\check{\varphi}$ is surjective.

In order to prove (d), we must establish that $id_{\mathcal{M}(S; \Sigma)} \circ \check{\varphi} = \check{\beta}_S \circ id_{\mathcal{M}(Y; \Sigma)}$. We show that it holds for a generator $(r, P, rf) \in \text{Ind}_{\mathcal{M}(S; \Sigma)}$ and note that it can be extended to all $(\text{Ind}_{\mathcal{M}(S; \Sigma)})^+$. Since $\check{\beta}_S$ agrees with $\check{\varphi}$ on $\text{Ind}_{\mathcal{M}(S; \Sigma)}$, we have that:

$$\begin{aligned} (r, P, rf)(id_{\mathcal{M}(S; \Sigma)} \circ \check{\varphi}) &= (r, P, rf)\check{\varphi} \\ &= (r, P, rf)\check{\beta}_S \\ &= (r, P, rf)\check{\beta}_S \circ id_{\mathcal{M}(Y; \Sigma)}. \end{aligned}$$

This shows that $\check{\varphi}, \check{\beta}_S$ is a semigroup system homomorphism. Since $\check{\varphi}$ is surjective, this implies that $\check{\beta}_S$ is also surjective. We conclude that $\check{\varphi}, \check{\beta}_S$ is a surjective semigroup system homomorphism. \blacksquare

We wish to show that our construction is a functor from Σ -generated semigroup systems to semigroup systems with generating sets. We will denote the map (between semigroup systems and semigroup system homomorphisms) by $\mathcal{M}(_; \Sigma)$.

Theorem 3.2.6. *The map $\mathcal{M}(_; \Sigma)$ is a functor from \mathbf{SGP}_Σ to $\mathbf{SGP-SYS}$. Moreover, $\mathcal{M}(_; \Sigma)$ is a semigroup expansion.*

Proof: The map $\mathcal{M}(_; \Sigma)$ sends a semigroup system (S, Σ, f) to the semigroup system $(\mathcal{M}(S; \Sigma), \text{Ind}_{\mathcal{M}(S; \Sigma)}, \text{id}_{\mathcal{M}(S; \Sigma)})$. Moreover it maps a Σ -preserving semigroup system homomorphism $\varphi, \text{id}_\Sigma : (S, \Sigma, f_S) \rightarrow (Y, \Sigma, f_Y)$ to a surjective semigroup system homomorphism $\check{\varphi}, \check{\beta}_S$. We must check that $\mathcal{M}(_; \Sigma)$ satisfies the appropriate functorial properties. To this end, let (S, Σ, f_S) , (Y, Σ, f_Y) , and (Z, Σ, f_Z) be semigroup systems and let $\varphi : S \rightarrow Y$ and $\delta : Y \rightarrow Z$ be Σ -preserving semigroup system homomorphisms. Lemma 3.2.5(c) establishes that φ and δ induce the following surjective semigroup system homomorphisms:

$$\check{\varphi}, \check{\beta}_S : (\mathcal{M}(S; \Sigma), \text{Ind}_{\mathcal{M}(S; \Sigma)}, \text{id}_{\mathcal{M}(S; \Sigma)}) \rightarrow (\mathcal{M}(Y; \Sigma), \text{Ind}_{\mathcal{M}(Y; \Sigma)}, \text{id}_{\mathcal{M}(Y; \Sigma)})$$

and

$$\check{\delta}, \check{\beta}_Y : (\mathcal{M}(Y; \Sigma), \text{Ind}_{\mathcal{M}(Y; \Sigma)}, \text{id}_{\mathcal{M}(Y; \Sigma)}) \rightarrow (\mathcal{M}(Z; \Sigma), \text{Ind}_{\mathcal{M}(Z; \Sigma)}, \text{id}_{\mathcal{M}(Z; \Sigma)}).$$

Given Proposition 3.2.1(e), we know that $\epsilon_S : (r, P, c) \rightarrow c$ maps $\mathcal{M}(S; \Sigma)$ to S . Using these results, it is a trivial check that the diagrams below commute.

We conclude that $\mathcal{M}(_; \Sigma)$ is a functor from \mathbf{SGP}_Σ to $\mathbf{SGP-SYS}$. It remains to show that $\mathcal{M}(_; \Sigma)$ is a semigroup expansion. To this end, consider a semigroup system (S, Σ, f) and the system for its graph expansion $(\mathcal{M}(S; \Sigma), \text{Ind}_{\mathcal{M}}, \text{id}_{\mathcal{M}})$. We

$$\begin{array}{ccc}
(\mathcal{M}(S; \Sigma), \text{Ind}_{\mathcal{M}} id_{\mathcal{M}}) & \xrightarrow{id_{\mathcal{M}(S; \Sigma)}, id_{\text{Ind}}} & (\mathcal{M}(S; \Sigma), \text{Ind}_{\mathcal{M}} id_{\mathcal{M}}) \\
\downarrow \epsilon_S, \epsilon_{\Sigma} & & \downarrow \epsilon_S, \epsilon_{\Sigma} \\
(S, \Sigma, f_S) & \xrightarrow{id_S, id_{\Sigma}} & (S, \Sigma, f_S)
\end{array}$$

$$\begin{array}{ccccc}
(\mathcal{M}(S; \Sigma), \text{Ind}_{\mathcal{M}} id_{\mathcal{M}}) & \xrightarrow{\check{\varphi}, \check{\beta}_S} & (\mathcal{M}(Y; \Sigma), \text{Ind}_{\mathcal{M}} id_{\mathcal{M}}) & \xrightarrow{\check{\delta}, \check{\beta}_Y} & (\mathcal{M}(Z; \Sigma), \text{Ind}_{\mathcal{M}} id_{\mathcal{M}}) \\
\downarrow \epsilon_S, \epsilon_{\Sigma} & & \downarrow \epsilon_Y, \epsilon_{\Sigma} & & \downarrow \epsilon_Z, \epsilon_{\Sigma} \\
(S, \Sigma, f_S) & \xrightarrow{\varphi, id_{\Sigma}} & (Y, \Sigma, f_Y) & \xrightarrow{\delta, id_{\Sigma}} & (Z, \Sigma, f_Z)
\end{array}$$

Figure 3.7: Diagrams showing that the graph expansion of semigroups satisfies the properties to be a functor.

showed in Proposition 3.2.4 that the maps $\epsilon_S, \epsilon_{\Sigma}$ form a surjective semigroup system homomorphism from $(\mathcal{M}(S; \Sigma), \text{Ind}_{\mathcal{M}}, id_{\mathcal{M}})$ to (S, Σ, f) . Moreover the function $\epsilon_{_, \epsilon_{\Sigma}}$ makes the diagrams shown in Figure 3.8 commute. Thus $\epsilon_{_, \epsilon_{\Sigma}}$ is a natural transformation between the functor $\mathcal{M}(_, \Sigma)$ and the identity functor on Σ -generated semigroup systems. This then implies that $\mathcal{M}(_, \Sigma)$ is a semigroup expansion. ■

In the category **SGP** $_{\Sigma}$

$$\begin{array}{c}
(S, \Sigma, f_S) \\
\downarrow \varphi, id_{\Sigma} \\
(Y, \Sigma, f_Y)
\end{array}$$

In the category **SGP-SYS**

$$\begin{array}{ccc}
(\mathcal{M}(S; \Sigma), \text{Ind}_{\mathcal{M}}, id_{\mathcal{M}}) & \xrightarrow{\epsilon_S, \epsilon_{\Sigma}} & (S, \Sigma, f_S) \\
\downarrow \check{\varphi}, \check{\beta} & & \downarrow \varphi, id_{\Sigma} \\
(\mathcal{M}(Y; \Sigma), \text{Ind}_{\mathcal{M}}, id_{\mathcal{M}}) & \xrightarrow{\epsilon_Y, \epsilon_{\Sigma}} & (Y, \Sigma, f_Y)
\end{array}$$

Figure 3.8: Diagram showing the natural transformation from the semigroup graph expansion functor to the semigroup system identity functor.

3.3 The Monoid and Semigroup Path Expansions

Monoid and semigroup graph expansions capture many aspects of the group graph expansion in their use of rooted, finite graphs with chosen vertices, and by their similar multiplication. However, they fail to incorporate two related aspects of the group construction: first if $(P, c) \in M_{gp}(G; \Omega)$, then there exists a word $w \in (\Omega \cup \Omega^{-1})^*$ that labels a path from 1 to c that traverses all edges of the digraph P ; second, $M_{gp}(G; \Omega)$ is generated (as an inverse monoid) by Ω . Motivated by this, we will introduce two expansions, the monoid path expansion and the semigroup path expansion, that have analogous properties.

The *path expansion of a monoid system* (T, Λ, f) , denoted $\text{Path}_{mon}(T; \Lambda)$, is defined:

$$\text{Path}_{mon}(T; \Lambda) = \left\{ (P, c) \left| \begin{array}{l} \text{there exists a word } w \in \Lambda^*, \text{ such that } wf = c \\ \text{and the path } 1 \xrightarrow{w} c \text{ is contained in } P \\ \text{and traverses every edge of } P \end{array} \right. \right\}$$

with multiplication defined as for the monoid graph expansion. The basic results about $\text{Path}_{mon}(T; \Lambda)$ are:

Proposition 3.3.1. *Let (T, Λ, f) be a monoid system.*

1. *then $\text{Path}_{mon}(T; \Lambda)$ is a submonoid of $M_{mon}(T; \Lambda)$;*
2. *T is the image of $\text{Path}(T; \Lambda)$ under ϵ_T ;*
3. *$\text{Path}_{mon}(T; \Lambda)$ is a Λ -generated monoid; it is generated by the set*

$$\{(\bullet_1 \xrightarrow{s} \bullet, sf) \mid s \in \Sigma\}.$$

Proof: Starting with (a), observe that the identity $(\bullet_1, 1) \in \text{Path}_{mon}(T; \Lambda)$. Next, suppose $(P, c), (Q, d) \in \text{Path}_{mon}(T; \Lambda)$. There exist words $v, w \in \Lambda^*$ such that $1 \xrightarrow{v} c$ is a path in P traversing every edge of P and $1 \xrightarrow{w} d$ is a path in Q traversing every edge of Q . The later path implies the existence of a path $c \xrightarrow{w} cd$ in cQ traversing every edge of cQ . Thus the path $1 \xrightarrow{vw} cd$ is in $P \cup cQ$ and traverses every

edge of $P \cup cQ$. Hence $(P, c)(Q, d) \in \text{Path}_{\text{mon}}(T; \Lambda)$, whereupon $\text{Path}_{\text{mon}}(T; \Lambda)$ is a submonoid.

Proceeding to (b), let $c \in T$. There exists some word $w \in \Lambda^+$ such that $wf = c$. Since $(\lfloor 1 \xrightarrow{w} c \rfloor, c) \in \text{Path}_{\text{mon}}(T; \Lambda)$ and $(\lfloor 1 \xrightarrow{w} c \rfloor, c)\epsilon_T = c$, we obtain that $\text{Path}_{\text{mon}}(T; \Lambda)\epsilon_T = T$.

Finally, for part (c), we let $(P, c) \in \text{Path}(S; \Sigma)$. Then there exists a word $w \in \Sigma^*$ that labels an $1 \rightarrow c$ path that traverses every edge of P . We write the word w as $w = w_1 w_2 \dots w_n$ where each $w_i \in \Sigma$. Thus

$$(P, c) = (\bullet_1 \xrightarrow{w_1} \bullet_{w_1 f}, w_1 f)(\bullet_1 \xrightarrow{w_2} \bullet_{w_2 f}, w_2 f) \dots (\bullet_1 \xrightarrow{w_n} \bullet_{w_n f}, w_n f).$$

We conclude that $\{(\bullet_1 \xrightarrow{s} \bullet_{sf}, sf) \mid s \in \Sigma\}$ generates $\text{Path}(S; \Sigma)$. ■

The monoid path expansion was previously introduced under the name ‘‘Cayley expansion’’ by Elston in [3]. She defines the *monoid Cayley expansion* of a monoid system (T, Λ, f) , denoted $\text{CayExp}_{\text{mon}}(T; \Lambda)$, as the submonoid of $M_{\text{mon}}(T; \Lambda)$ that is generated by the set $\{(\bullet_1 \xrightarrow{sf} \bullet_{sf}, sf) \mid s \in \Lambda\}$. From Proposition 3.3.1 (3), it is clear that $\text{Path}_{\text{mon}}(T; \Lambda) = \text{CayExp}_{\text{mon}}(T; \Lambda)$.

For Elston, the monoid path expansion is an example of a more general approach to constructing expansions using derived categories. Thus, she does not study the construction in depth. She does provide an alternative characterization of the monoid path expansion. To state Elston’s result, we need the following definitions: given a surjective semigroup homomorphism $\varphi : S \rightarrow T$, the *derived category* D_φ is the category whose objects are the elements of T and whose arrows are triples of the form $[t_1, s, t_2]$ where $t_1, t_2 \in T$ and $s \in S$ satisfy the equation $t_1(s\varphi) = t_2$. Two arrows $[t_1, s, t_2]$ and $[t'_1, s', t'_2]$ are considered equal if $t_1 = t'_1$, $t_2 = t'_2$, and for every $s_0 \in t_1\varphi^{-1}$, we have $s_0 s = s_0 s'$. Multiplication of arrows $[t_1, s, t_2]$ and $[t_3, s', t_4]$ is defined if $t_2 = t_3$, in which case the product is $[t_1, ss', t_4]$. For each $t \in T$, the local semigroup of the derived category at t is the subset of the form $\{[t, s, t] \mid [t, s, t] \in D_\varphi\}$. If S is a monoid, then all local semigroups are local monoids.

Theorem 3.3.2. [Elston, [3]] *Let (T, Λ, f) be a monoid system. The monoid path expansion $\text{Path}_{\text{mon}}(T; \Lambda)$ is the largest monoid expansion in which the local monoids of the derived category of the homomorphism from the expansion to T are semilattices.*

Elston then uses the monoid Cayley expansion to create a semigroup version: the *semigroup Cayley expansion* of a semigroup system (S, Σ, f) , denoted $\text{CayExp}_{\text{sgp}}(S; \Sigma)$, is the subsemigroup of $\text{CayExp}_{\text{mon}}(S^1; \Sigma)$ generated by $\{(\bullet \xrightarrow[sf]{sf} \bullet, sf) \mid s \in \Sigma\}$. It also can be characterized using derived categories:

Theorem 3.3.3. [Elston, [3]] *Let (S, Σ, f) be a semigroup system. The semigroup Cayley expansion is the largest semigroup expansion in which the local semigroups of the derived category of the homomorphism from the expansion to S are semilattices.*

In contrast to the case for monoids, Elston's semigroup Cayley expansion is not a subsemigroup of the semigroup graph expansion. Thus we now present a construction that resides inside the semigroup graph expansion: the *path expansion of a semigroup system* (S, Σ, f) , denoted by $\text{Path}(S; \Sigma)$, is the set of triples:

$$\text{Path}(S; \Sigma) = \left\{ (r, P, c) \left| \begin{array}{l} \text{there exists a word } w \in \Sigma^*, \text{ such that } (rw)f = c \\ \text{and the path } rf \xrightarrow{w} c \text{ is contained in } P \\ \text{and traverses every edge of } P \end{array} \right. \right\}$$

with multiplication defined as for the semigroup graph expansion. In Figure 3.9, we show two elements of $\mathcal{M}(S; \{r, s, t\})$ where $(S, \{r, s, t\}, id)$ is the right zero semigroup system introduced at the start of Section 3.2 (and shown in Figure 3.1). The element (s, Q, s) is not in $\text{Path}(S; \{r, s, t\})$; in contrast, the element (s, Q, r) is in $\text{Path}(S; \{r, s, t\})$ since $s \xrightarrow{sr} r$ traverses all the edges of Q .

The semigroup path expansion and Elston's semigroup Cayley expansion are not equivalent constructions. We demonstrate this through an example.

Example: Consider the semigroup system $(S, \{x\}, id)$ for the semigroup presented $S = \langle x \mid x = x^3 \rangle$. This semigroup has an identity, the element x^2 . Thus $S^1 = S$.

(s, Q, s)

(s, Q, r)



Figure 3.9: The element (s, Q, r) is in $\text{Path}(S; \{r, s, t\})$, but (s, Q, s) is not.

Noting that $x^2 = 1$, the semigroup Cayley expansion $\text{CayExp}_{sgp}(S; \{x\})$ consists of three elements:

$$\left\{ \left(\begin{array}{c} \bullet \xrightarrow{x^2} \odot_x \\ x^2 \end{array}, x \right), \left(\begin{array}{c} \odot_{x^2} \xleftarrow{x} \bullet_x \\ x^2 \end{array}, x^2 \right), \left(\begin{array}{c} \bullet_{x^2} \xrightarrow{x} \odot_x \\ x^2 \end{array}, x \right) \right\},$$

whereas the semigroup path expansion $\text{Path}(S; \{x\})$ contains four elements:

$$\left\{ \left(x, \begin{array}{c} \odot_x \\ x \end{array}, x \right), \left(x, \begin{array}{c} \odot_{x^2} \xleftarrow{x} \bullet_x \\ x^2 \end{array}, x^2 \right), \left(x, \begin{array}{c} \bullet_{x^2} \xrightarrow{x} \odot_x \\ x^2 \end{array}, x \right), \left(x, \begin{array}{c} \odot_{x^2} \xrightarrow{x} \bullet_x \\ x^2 \end{array}, x^2 \right) \right\}.$$

Additionally, there are elements of the local monoids of the derived category of $\epsilon_S : \text{Path}_S \rightarrow S$ that are not idempotents. For example, from the equation

$$\left(x, \begin{array}{c} \odot_x \\ x \end{array}, x \right) \left(x, \begin{array}{c} \odot_{x^2} \xleftarrow{x} \bullet_x \\ x^2 \end{array}, x^2 \right) = \left(x, \begin{array}{c} \bullet_{x^2} \xrightarrow{x} \odot_x \\ x^2 \end{array}, x \right),$$

we see that $[x, (x, \bullet_{x^2} \xleftarrow{x} \bullet_x, x^2), x]$ is in the local monoid at x . However, it does not equal its square and thus is not idempotent:

$$\left[x, \left(x, \begin{array}{c} \odot_{x^2} \xleftarrow{x} \bullet_x \\ x^2 \end{array}, x^2 \right), x \right]^2 = \left[x, \left(x, \begin{array}{c} \odot_{x^2} \xrightarrow{x} \bullet_x \\ x^2 \end{array}, x^2 \right), x \right].$$

Hence we see that the local monoids of the derived category of the semigroup path expansion are not always semilattices. ■

We now establish some basic results about $\text{Path}(S; \Sigma)$:

Proposition 3.3.4. *Let (S, Σ, f) be a semigroup system. Then*

- (a) $\text{Path}(S; \Sigma)$ is a subsemigroup of $\mathcal{M}(S; \Sigma)$;
- (b) $\text{Ind}_{\text{Path}(S; \Sigma)} = \{(s, \bullet_{sf}, sf) \mid s \in \Sigma\}$;
- (c) the image of $\text{Path}(S; \Sigma)$ under ϵ_S is S ;
- (d) $(\text{Path}(S; \Sigma), \Sigma, g)$ where g is given by $r \mapsto (r, \bullet_{rf}, rf)$ is a semigroup system; the functions ϵ_S, id_Σ form a surjective semigroup system homomorphism from $(\text{Path}(S; \Sigma), \Sigma, g)$ to (S, Σ, f) .

Proof: In order to show (a), suppose $(r, P, c), (s, Q, d) \in \text{Path}(S; \Sigma)$. There exist words $v, w \in \Sigma^*$ such that $rf \xrightarrow{v} c$ is a path in P traversing every edge of P and $sf \xrightarrow{w} d$ is a path in Q traversing every edge of Q . The later path implies the existence of a path $c \xrightarrow{sw} cd$ in cQ_s^1 traversing every edge of cQ_s^1 . Thus the path $rf \xrightarrow{vsw} cd$ is in $P \cup cQ_s^1$ and traverses every edge of $P \cup cQ_s^1$. Hence $(r, P, c)(s, Q, d) \in \text{Path}(S; \Sigma)$, whereupon $\text{Path}(S; \Sigma)$ is a subsemigroup.

Part (b) follows from the description of indecomposable elements given in Lemma 3.2.2(b) and the fact that if $(r, P, c) \in \text{Path}(S; \Sigma)$, then there is an edge in P entering c except when $(r, P, c) = (r, \bullet_{rf}, rf)$.

Moving on to a proof of (c), let $c \in S$. There exists some $w \in \Sigma^+$ such that $wf = c$. Write $w = rv$, where $r \in \Sigma, v \in \Sigma^*$. Since $(r, [rf \xrightarrow{v} c], c) \in \text{Path}(S; \Sigma)$ and $(r, [rf \xrightarrow{v} c], c)\epsilon_S = c$, we see that $(\text{Path}(S; \Sigma))\epsilon_S = S$.

We now show (d). Since the indecomposable elements of a semigroup generate the semigroup, it follows from part (b) that $(\text{Path}(S; \Sigma), \Sigma, g)$ is a semigroup system. Now we wish to show that the maps ϵ_S, id_Σ form a semigroup system homomorphism from $(\text{Path}(S; \Sigma), \Sigma, g)$ to (S, Σ, f) . First we must show that $g \circ \epsilon_S = id_\Sigma \circ f$. Let $r \in \Sigma$. Then we have:

$$\begin{aligned}
r(g \circ \epsilon_S) &= (r, \bullet_{rf}, rf)\epsilon_S \\
&= rf \\
&= r(id_\Sigma \circ f)
\end{aligned}$$

We conclude that the maps ϵ_S, id_Σ together constitute a semigroup system homomorphism. Having shown in part (c) that the image of ϵ_S is S and noting that the identity map id_Σ is clearly surjective on Σ^+ , we know that ϵ_S, id_Σ is a surjective semigroup system homomorphism. ■

We conclude this section by proving that the semigroup path expansion is a functor, which we denote by $\text{Path}(_; \Sigma)$, from the category \mathbf{SGP}_Σ to itself. We will also show that $\text{Path}(S; \Sigma)$ is an expansion. The following lemma will be useful:

Lemma 3.3.5. *Let (S, Σ, f_S) and (Y, Σ, f_Y) be semigroup systems and let $\varphi : S \rightarrow Y$ be a Σ -preserving semigroup system homomorphism. Then $\check{\varphi}$ restricted to $\text{Path}(S; \Sigma)$ is a Σ -preserving semigroup system homomorphism with image $\text{Path}(Y; \Sigma)$.*

Proof: The path expansion sends the semigroup systems (S, Σ, f_S) and (Y, Σ, f_Y) to $(\text{Path}(S; \Sigma), \Sigma, g_S)$ and $(\text{Path}(Y; \Sigma), \Sigma, g_Y)$ respectively. We showed in Lemma 3.2.5 (c) that $\check{\varphi}$ is a semigroup system homomorphism between $\mathcal{M}(S; \Sigma)$ and $\mathcal{M}(Y; \Sigma)$. Hence the restriction of $\check{\varphi}$ to $\text{Path}(S; \Sigma)$ is also a homomorphism. To see that its image is $\text{Path}(Y; \Sigma)$, suppose that $(s, Q, d) \in \text{Path}(Y; \Sigma)$. Then there exists a word $w \in \Sigma^+$ such that $[sf_Y \xrightarrow{w} d] = Q$. The element $(s, [sf_S \xrightarrow{w} (sw)f_S], (sw)f_S)$ is in $\text{Path}(S; \Sigma)$. Moreover we have that $(s, [sf_S \xrightarrow{w} (sw)f_S], (sw)f_S)\check{\varphi} = (s, Q, d)$. Thus $\text{Path}(S; \Sigma)\check{\varphi} = \text{Path}(Y; \Sigma)$. Next we prove that the restriction of $\check{\varphi}$ is Σ -preserving. To do so, we must show that $g_S \circ \check{\varphi} = g_Y$. Let $t \in \Sigma$. Then it follows

$$\begin{aligned}
t(g_S \circ \check{\varphi}) &= (t, \bullet_{tf_S}, tf_S)\check{\varphi} \\
&= (t, \bullet_{tf_S} \widehat{\varphi}, tf_S \circ \varphi) \\
&= (t, \bullet_{tf_S \circ \varphi}, tf_Y) \\
&= (t, \bullet_{tf_Y}, tf_Y) \\
&= tg_Y.
\end{aligned}$$

We see that $g_S \circ \check{\varphi} = g_Y$. We conclude that $\check{\varphi}$ is Σ -preserving. ■

$$\begin{array}{ccccc}
(\text{Path}(S; \Sigma), \Sigma, g_S) & \xrightarrow{id_{\text{Path}}, id_\Sigma} & (\text{Path}(S; \Sigma), \Sigma, g_S) & & \\
\downarrow \epsilon_S, id_\Sigma & & \downarrow \epsilon_S, id_\Sigma & & \\
(S, \Sigma, f_S) & \xrightarrow{id_S, id_\Sigma} & (S, \Sigma, f_S) & & \\
\\
(\text{Path}(S; \Sigma), \Sigma, g_S) & \xrightarrow{\check{\varphi}, id_\Sigma} & (\text{Path}(Y; \Sigma), \Sigma, g_Y) & \xrightarrow{\check{\delta}, id_\Sigma} & (\text{Path}(Z; \Sigma), \Sigma, g_Z) \\
\downarrow \epsilon_S, id_\Sigma & & \downarrow \epsilon_Y, id_\Sigma & & \downarrow \epsilon_Z, id_\Sigma \\
(S, \Sigma, f_S) & \xrightarrow{\varphi, id_\Sigma} & (Y, \Sigma, f_Y) & \xrightarrow{\delta, id_\Sigma} & (Z, \Sigma, f_Z)
\end{array}$$

Figure 3.10: Diagram showing that the path expansion of semigroups is a functor.

Theorem 3.3.6. *The map $\text{Path}(_; \Sigma)$ is a functor from \mathbf{SGP}_Σ to itself. Moreover, $\text{Path}(_; \Sigma)$ is a semigroup expansion.*

Proof: The map $\text{Path}(_; \Sigma)$ sends a Σ -generated semigroup system (S, Σ, f) to the Σ -generated semigroup system $(\text{Path}(S; \Sigma), \Sigma, g)$. Moreover it maps a Σ -preserving semigroup system homomorphism $\varphi, id_\Sigma : (S, \Sigma, f_S) \rightarrow (Y, \Sigma, f_Y)$ to a surjective semigroup system homomorphism $\check{\varphi}, id_\Sigma$. We must check that $\text{Path}(_; \Sigma)$ satisfies the appropriate functorial properties. To this end, let (S, Σ, f_S) , (Y, Σ, f_Y) , and (Z, Σ, f_Z) be semigroup systems and $\varphi : S \rightarrow Y$ and $\delta : Y \rightarrow Z$ be Σ -preserving semigroup system homomorphisms. It is straightforward to check that the diagrams in Figure 3.10 commute. Thus $\text{Path}(_; \Sigma)$ is a functor from \mathbf{SGP}_Σ to itself.

We will also show that $\text{Path}(_; \Sigma)$ is a semigroup expansion. Given a semigroup system (S, Σ, f) and its path expansion $(\text{Path}(S; \Sigma), \Sigma, id)$. We showed in Proposition 3.3.4(d) that the maps ϵ_S, id_Σ form a surjective semigroup system homomorphism from $(\text{Path}(S; \Sigma), \Sigma, id)$ to (S, Σ, f) . Moreover the function ϵ_S, id_Σ makes the diagrams shown in Figure 3.11 on page 46 commute. Thus ϵ_S, id_Σ is a natural trans-

$$\begin{array}{ccc}
(\text{Path}(\mathbf{S}; \Sigma), \Sigma, g_S) & \xrightarrow{\varepsilon_S, id_\Sigma} & (\mathbf{S}, \Sigma, f_S) \\
\downarrow \check{\varphi}, id_\Sigma & & \downarrow \varphi, id_\Sigma \\
(\text{Path}(\mathbf{Y}; \Sigma), \Sigma, g_Y) & \xrightarrow{\varepsilon_Y, id_\Sigma} & (\mathbf{Y}, \Sigma, f_Y)
\end{array}$$

Figure 3.11: Diagram showing the natural transformation from the semigroup path expansion functor to the semigroup system identity functor.

formation between the functor $\text{Path}(_; \Sigma)$ and the identity functor on Σ -generated semigroup systems. This then implies that $\text{Path}(_; \Sigma)$ is a semigroup expansion. ■

Similarly, the monoid path expansion is a functor from \mathbf{MON}_Λ to itself. We denote this functor by $\text{Path}_{mon}(_; \Lambda)$.

Chapter 4

Examples and Example-Specific Properties

We devote this section to looking at the properties of $\mathcal{M}(S; \Sigma)$ for various cases of semigroups: namely free semigroups, groups presented as semigroups, left-zero and right-zero semigroups, direct products with one factor that is a left-zero semigroup (in particular rectangular bands), and semilattices. These examples suggest the range of characteristics that graph expansions of semigroups can have.

4.1 Free Semigroups

We would like to describe graph expansions of semigroup systems of free semigroups. For simplicity, we will restrict ourselves to free semigroups generated in a very straightforward manner. Namely, we look at semigroup systems of the form (S, Σ, id) . Note that for this case, $S \cong \Sigma^+$. This restriction ensures that in the Cayley digraph $\text{Cay}(S; \Sigma)$, there is exactly one edge entering each vertex. We now characterize the elements of the graph expansion:

Proposition 4.1.1. *Let $(S; \Sigma, id)$ be a semigroup system of a free semigroup S . If $(r, P, c) \in \mathcal{M}(S; \Sigma)$, then P is an rf-rooted tree containing the vertex c .*

Proof: The definition of the semigroup graph expansion implies that P is rf -rooted and contains c . Since the Cayley digraph $\text{Cay}(S; \Sigma)$ is comprised of $|\Sigma|$ trees and P is a rooted subdigraph, P is also a tree. ■

In Section 3.2, we used examples of elements from the graph expansion of a free semigroup, namely $(S, \{x, y\}, id)$, where S is a free semigroup, to illustrate the graph expansion operation. There are found in Figures 3.5 and 3.6 on page 28. We now turn to look at the free monogenic semigroup, the free semigroup which can be generated by one element. For this case, we will show that the graph expansion can be embedded into a semidirect product whose structure is very easy to understand. To this end, let $(S, \{x\}, id)$ be the semigroup system of a free monogenic semigroup S . The associated Cayley digraph $\text{Cay}(S; \{x\})$ is:

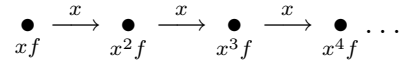


Figure 4.1: The Cayley digraph $\text{Cay}(S; \{x\})$ for a free monogenic semigroup S .

Take $\mathbb{N} = \{1, 2, 3, \dots\}$ and let \mathcal{N} be the semigroup (\mathbb{N}, \max) . We define an action of $(\mathbb{N}, +)$ on \mathcal{N} by $n \triangleright p = n + p$. This is a semigroup action since

$$m \triangleright n \triangleright p = m \triangleright (n + p) = m + n + p = (m + n) \triangleright p.$$

Moreover each $n \in \mathbb{N}$ also acts on \mathcal{N} :

$$n \triangleright \max(p, q) = n + \max(p, q) = \max(n + p, n + q) = \max(n \triangleright p, n \triangleright q).$$

This structure enables us to form the semidirect product $\mathcal{N} \rtimes \mathbb{N}$ with the binary

operation:

$$(p, m)(q, n) = (\max(p, m \triangleright q), m + n) = (\max(p, m + q), m + n).$$

The aim of the next proposition is to show that the graph expansion of a free monogenic semigroup is isomorphic to a subsemigroup of $\mathcal{N} \rtimes \mathbb{N}$. Put another way, since $(\mathbb{N}, +)$ is isomorphic to the free monogenic semigroup x^* , what we are doing is showing that $\mathcal{M}(x^*, \{x\})$ embeds in $\mathcal{N} \rtimes x^*$.

Proposition 4.1.2. *Consider the semigroup system $(S, \{x\}, id)$ for the free monogenic semigroup S . Then $\mathcal{M}(S; \{x\})$ embeds in $\mathcal{N} \rtimes \mathbb{N}$ as the subsemigroup $\{(p, m) \mid p \geq m\}$.*

Proof: All elements of $\mathcal{M}(S; \{x\})$ have the form $(x, \lfloor x \xrightarrow{p-1} x^p \rfloor, x^c)$ where $c \leq p$. Let $\alpha : \mathcal{M}(S; \{x\}) \rightarrow \{(p, m) \mid p \geq m\}$ be the map given by $(x, \lfloor x \xrightarrow{p-1} x^p \rfloor, x^c) \mapsto (p, c)$. Further, let $(x, \lfloor x \xrightarrow{p-1} x^p \rfloor, x^c), (x, \lfloor x \xrightarrow{q-1} x^q \rfloor, x^d) \in \mathcal{M}(S; \{x\})$. We show that α is a homomorphism:

$$\begin{aligned} & ((x, \lfloor x \xrightarrow{p-1} x^p \rfloor, x^c)(x, \lfloor x \xrightarrow{q-1} x^q \rfloor, x^d))\alpha \\ &= (x, \lfloor x \xrightarrow{m-1} x^m \rfloor, x^{c+d})\alpha \quad \text{where } m = \max\{p, c+q\} \\ &= (m, c+d) \\ &= (p, c)(q, d) \\ &= (x, \lfloor x \xrightarrow{p-1} x^p \rfloor, x^c)\alpha(x, \lfloor x \xrightarrow{q-1} x^q \rfloor, x^d)\alpha. \end{aligned}$$

Thus the map α is a homomorphism. It is easy to check that it is bijective. This establishes the isomorphism between $\mathcal{M}(S; \{x\})$ and the subsemigroup $\{(p, m) \mid p \geq m\}$. ■

The Path Expansion for Free Semigroups

Looking at the diagram like shown in Figure 4.1 on page 48, it is a straightforward observation that if S is a free monogenic semigroup and $|\Sigma| = 1$, then $\text{Path}(S; \Sigma)$ is isomorphic to S . This result can be extended to describe all path expansions of free

semigroups. Moreover, it can be further extended to all path expansions of semigroups whose Cayley digraphs do not contain any cycles. A semigroup S will have a Cayley digraph that does not contain cycles precisely when for all $x \in S$, we have $x \notin xS$. The following fact about such semigroups will be useful:

Lemma 4.1.3. *Let (S, Σ, f) be a semigroup system of a semigroup S with the property that for all $x \in S$, we have that $x \notin xS$. If $x \xrightarrow{v} y$ and $x \xrightarrow{w} y$ are paths in $\text{Cay}(S; \Sigma)$ such that $\lfloor x \xrightarrow{v} y \rfloor = \lfloor x \xrightarrow{w} y \rfloor$, then $v = w$.*

Proof: Assume the semigroup S has the property that for all $x \in S$, we have that $x \notin xS$. Let $x \xrightarrow{v} y$ and $x \xrightarrow{w} y$ be paths in $\text{Cay}(S; \Sigma)$ such that $\lfloor x \xrightarrow{v} y \rfloor = \lfloor x \xrightarrow{w} y \rfloor$. Write $v = v_1 v_2 \dots v_m$ and $w = w_1 w_2 \dots w_n$ where each $v_i, w_i \in \Sigma$. By way of contradiction, assume there is a value k such that $v_k \neq w_k$. However, as the underlying edge sets are the same, the path $x \xrightarrow{v} y$ passes through the edge $(x(v_1 v_2 \dots v_{k-1})f, w_k)$ and thus there must be some prefix v' of v distinct from $v_1 v_2 \dots v_{k-1}$ such that $x(v'f) = x(v_1 v_2 \dots v_{k-1})f$. Write $v' = v_1 v_2 \dots v_j$, again with each $v_i \in \Sigma$. Note that $k - 1 \neq j$.

If $j < k - 1$, then $x(v_1 \dots v_j)f = x(v_1 \dots v_j)f(v_{j+1} \dots v_{k-1})f$. On the other hand, if $j > k - 1$, then $(v_1 \dots v_{k-1})f = (v_1 \dots v_{k-1})f(v_k \dots v_j)f$. In both situations, there is an element x for which $x \in xS$. As we assumed that there is no such element with this property, we conclude that $v = w$. \blacksquare

We can now give the main result. Recall that Σ^+ is isomorphic to any free semigroup on $|\Sigma|$ generators.

Proposition 4.1.4. *Let (S, Σ, f) be a semigroup system of a semigroup S with the property that for all $x \in S$, we have that $x \notin xS$. Then $\text{Path}(S; \Sigma) \cong \Sigma^+$.*

Proof:

Define a map $\alpha : \Sigma^+ \rightarrow \text{Path}(S; \Sigma)$ by

$$\begin{aligned} w &\mapsto (w, \bullet_{wf}, wf) && \text{if } w \in \Sigma; \\ w &\mapsto (r, \lfloor rf \xrightarrow{v} wf \rfloor, wf) && \text{if } w = rv \text{ with } r \in \Sigma \text{ and } v \in \Sigma^+. \end{aligned}$$

To see that α is a homomorphism, let $w_1, w_2 \in \Sigma^+$. There are four cases, depending upon whether v and w are of length one or greater. We show the case $w_1, w_2 \in \Sigma$, and note that the others are similar:

$$\begin{aligned}
(w_1 w_2) \alpha &= (w_1, [w_1 f \xrightarrow{w_2} (w_1 w_2) f], (w_1 w_2) f) \\
&= (w_1, \bullet_{w_1 f}, w_1 f) (w_2, \bullet_{w_2 f}, w_2 f) \\
&= (w_1 \alpha) (w_2 \alpha).
\end{aligned}$$

In order to see that α is injective, consider $w_1, w_2 \in \Sigma^+$ such that $w_1 \alpha = w_2 \alpha$. We rewrite the words as $w_1 = r_1 v_1$ and $w_2 = r_2 v_2$ where $r_1, r_2 \in \Sigma$ and $v_1, v_2 \in \Sigma^*$. This is the same as $(r_1, [r_1 f \xrightarrow{v_1} w_1 f], w_1 f) = (r_2, [r_2 f \xrightarrow{v_2} w_2 f], w_2 f)$, from which it follows that $r_1 = r_2$. Moreover, Lemma 4.1.3 guarantees that $v_1 = v_2$. This shows that α is injective.

To show that α is surjective, let $(r, [r f \xrightarrow{v} (rv) f], (rv) f) \in \text{Path}(S; \Sigma)$. Then $(rv) \alpha = (r, [r f \xrightarrow{v} (rv) f], (rv) f)$. Thus $\text{Path}(S; \Sigma) \cong \Sigma^+$. ■

Free semigroups are examples of semigroups which have the property that $x \notin xS$ and thus Proposition 4.1.4 describes their path expansion structure.

4.2 Semigroup Systems of Groups

We now investigate groups generated as semigroups. For this section, we assume (S, Σ, f) is a semigroup system of a group S . Though we do not know if $\text{Cay}(S; \Sigma)$ is a graph (this depends on the system), we do know that it is strongly connected and that for all $r \in \Sigma$ and $c \in S$, there is exactly one r -labeled edge terminating at c . These facts will aid us in showing that the graph expansion $\mathcal{M}(S; \Sigma)$ has the following characteristics.

Proposition 4.2.1. *Let (S, Σ, f) be a semigroup system of a group S . The following are true about $\mathcal{M}(S; \Sigma)$:*

- (a) an element (r, P, c) is idempotent if and only if $c = 1$ and $P = P_r^1$ (note that $P = P_r^1$ is equivalent to $(1, r) \in E(P)$);
- (b) $E(\mathcal{M}(S; \Sigma))$ is a subsemigroup;
- (c) let $(r, P, 1), (s, Q, 1) \in E(\mathcal{M}(S; \Sigma))$; then $(r, P, 1)$ and $(s, Q, 1)$ commute if and only if $r = s$;
- (d) fix $r \in \Sigma$; the set $\{(r, P, 1) \mid (r, P, 1) \text{ is idempotent}\}$ is a semilattice;
- (e) an element (r, P, c) is regular if and only if $P = P_r^1$ and P contains a path $c \longrightarrow 1$;
- (f) if (r, P, c) is regular, then (s, Q, d) is an inverse if and only if $d = c^{-1}$, $Q = c^{-1}P$, and there exists a path $c \longrightarrow 1$ in P which has (c, s) as its first edge;
- (g) $\text{Reg}(\mathcal{M}(S; \Sigma))$ is a subsemigroup; if $|\Sigma| = 1$, then $\text{Reg}(\mathcal{M}(S; \Sigma)) \cong S$; if $|\Sigma| \geq 2$, then $\text{Reg}(\mathcal{M}(S; \Sigma))$ is not an inverse semigroup, but is locally an inverse monoid.

Proof: Result (a) follows from Theorem 3.2.1(b) and the fact that 1 is the only idempotent in a group. To prove part (b), let $(r, P, 1), (s, Q, 1) \in E(\mathcal{M}(S; \Sigma))$. From (a), $P = P_r^1$, whereupon

$$\begin{aligned} P \cup cQ_s^1 &= P_r^1 \cup cQ_s^1 \\ &= (P \cup cQ_s^1)_r^1. \end{aligned}$$

Appealing to (a) again, the product $(r, P, 1)(s, Q, 1) = (r, P \cup cQ_s^1, 1)$ is idempotent. This gives the result (b).

Moving on to part (c), if idempotents $(r, P, 1)$ and $(s, Q, 1)$ commute, then Theorem 3.2.1(d) says that $r = s$. Conversely, suppose $r = s$. From (a), $P = P_r^1$ and $Q = Q_r^1$, whereupon $P \cup 1Q_r^1 = Q \cup 1P_r^1$. Thus,

$$(r, P, 1)(r, Q, 1) = (r, P \cup 1Q_r^1, 1) = (r, Q \cup 1P_r^1, 1) = (r, Q, 1)(r, P, 1).$$

Part (d) follows from (c).

We now wish to prove part (e). Suppose (r, P, c) is a regular element. From Theorem 3.2.1(c), there exists some $x \in V(P)$ such that there is a $c \rightarrow x$ path in P , $xc = c$, and $xP_r^1 \subseteq P$. In the group setting, the only candidate is $x = 1$, whereupon we know that $P = P_r^1$ and P contains a $c \rightarrow 1$ path. The converse is a straightforward consequence of Theorem 3.2.1(c).

In order to show part (f), assume (r, P, c) is regular and has inverse (s, Q, d) . Examining the third coordinate, we have that $cdc = c$. However, as multiplication for the third coordinate is in a group setting, this is possible only if $d = c^{-1}$. Thus we have

$$(r, P, c)(s, Q, c^{-1})(r, P, c) = (r, P \cup cQ_s^1 \cup P_r^1, c) = (r, P, c). \quad (4.2.1)$$

Similarly,

$$(s, Q, c^{-1})(r, P, c)(s, Q, c^{-1}) = (s, Q \cup c^{-1}P_r^1 \cup Q_s^1, c^{-1}) = (s, Q, c^{-1}). \quad (4.2.2)$$

From Equation 4.2.1, $cQ_s^1 \subseteq P$, whereupon $Q_s^1 \subseteq c^{-1}P$. Similarly, from Equation 4.2.2 we have that $c^{-1}P_r^1 \subseteq Q \subseteq Q_s^1$. Thus $Q = Q_s^1 = c^{-1}P$. Since Q is rooted at s , there is a $1 \rightarrow c^{-1}$ path with first edge $(1, s)$ in Q_s^1 . Translating both this path and Q_s^1 by c shows that there is a $c \rightarrow 1$ path with first edge (c, s) in $P = cQ_s^1$.

For the converse, let (r, P, c) be regular and $s \in \Sigma$ be the label of the first edge of a $c \rightarrow 1$ path in P . We want to show that the element $(s, c^{-1}P, c^{-1})$ is an inverse of (r, P, c) . Using the fact that $P = P_r^1$ and $(c, s) \in E(P)$, we have that

$$\begin{aligned} (r, P, c)(s, c^{-1}P, c^{-1})(r, P, c) &= (r, P \cup c(c^{-1}P)_s^1 \cup cc^{-1}P_r^1, cc^{-1}c) \\ &= (r, P \cup cc^{-1}P \cup \{(c, s)\} \cup P_r^1, c) \\ &= (r, P, c). \end{aligned}$$

The equation

$$(s, c^{-1}P, c^{-1})(r, P, c)(s, c^{-1}P, c^{-1}) = (s, c^{-1}P, c^{-1})$$

can be shown to hold for similar reasons.

Finally, we wish to prove part (g). Let (r, P, c) and (s, Q, d) be regular elements and consider the product $(r, P, c)(s, Q, d) = (r, P \cup cQ_s^1, cd)$. From (e), we know that $P = P_r^1$, which then implies that

$$P \cup cQ_s^1 = P_r^1 \cup cQ_s^1 = (P \cup cQ_s^1)_r^1.$$

Again using the characterization from (e), P contains a $c \rightarrow 1$ path and Q contains a $d \rightarrow 1$ path. Hence cQ_s^1 contains a $cd \rightarrow c$ path. Combining this information, $P \cup cQ_s^1$ contains a $cd \rightarrow 1$ path, whereupon, again by (e), $(r, P, c)(s, Q, d)$ is regular and we conclude that $\text{Reg}(\mathcal{M}(S; \Sigma))$ is a subsemigroup. Figure 4.2 illustrates the product of regular elements.

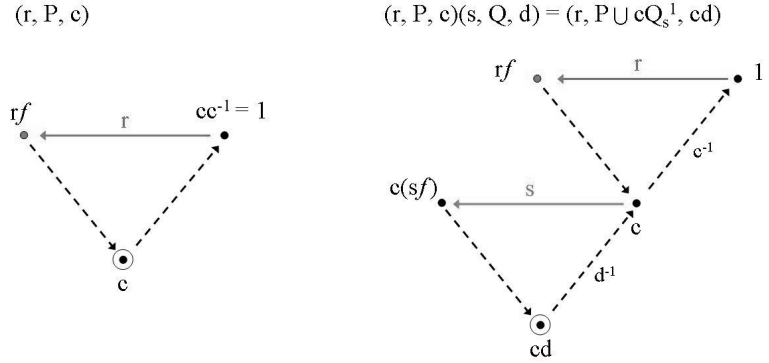


Figure 4.2: The diagram on the left shows a subdigraph that must be contained in P in order for an element (r, P, c) to be regular. It indicates that P must contain $rf \rightarrow c$ and $c \rightarrow rf$ paths. In the diagram on the right, we show a subdigraph that the product of regular elements will contain.

Suppose $|\Sigma| = \{r\}$. Then S is a finite cyclic group, since this is the only type

of group that can be generated as a semigroup by one element. Moreover, any path from rf to 1 passes through all edges of $\text{Cay}(S; \Sigma)$ except possibly $(1, r)$. If $(r, P, c) \in \text{Reg}(\mathcal{M}(S; \Sigma))$, then from (e), $P = P_r^1$ and P contains the path $c \longrightarrow 1$. Together these imply that $P = \text{Cay}(S; \Sigma)$. Thus there is exactly one regular element corresponding to each vertex $c \in S$. The map σ_S , when restricted to $\text{Reg}(\mathcal{M}(S; \Sigma))$, is an isomorphism between $\text{Reg}(\mathcal{M}(S; \Sigma))$ and S .

Suppose $|\Sigma| \geq 2$. We will construct two idempotents that do not commute. Let r and s be distinct elements of Σ . We can find words $u, w \in \Sigma^+$ such that $(rf)^{-1} = uf$ and $(sf)^{-1} = wf$. Then from (a), the elements $(r, [rf \xrightarrow{ur} rf], 1)$ and $(s, [sf \xrightarrow{ws} sf], 1)$ are idempotent. However from c, they do not commute and therefore $\text{Reg}(\mathcal{M}(S; \Sigma))$ is not an inverse semigroup.

Finally we show that $\text{Reg}(\mathcal{M}(S; \Sigma))$ is locally inverse. We will use the characterization of inverse semigroups that a regular semigroup is inverse if and only if its subset of idempotents commutes. Let $(r, P, 1) \in E(\mathcal{M}(S; \Sigma))$. We consider the subset $(r, P, 1)\text{Reg}(\mathcal{M}(S; \Sigma))(r, P, 1)$. Since it inherits closure from $\text{Reg}(\mathcal{M}(S; \Sigma))$, it is certainly a subsemigroup.

Let $(s, Q, d) \in \text{Reg}(\mathcal{M}(S; \Sigma))$. We claim that $(r, P, 1)(s, Q, d)(r, P, 1)$ has an inverse in $(r, P, 1)\text{Reg}(\mathcal{M}(S; \Sigma))(r, P, 1)$. Since (s, Q, d) is regular, by (f) we know that it has an inverse in $\mathcal{M}(S; \Sigma)$ of the form $(t, d^{-1}Q, d^{-1})$ where Q contains the edge (d, t) . Consider the product:

$$\begin{aligned}
& ((r, P, 1)(s, Q, d)(r, P, 1))((r, P, 1)(t, d^{-1}Q, d^{-1})(r, P, 1))((r, P, 1)(s, Q, d)(r, P, 1)) \\
&= (r, P, 1)(s, Q, d)(r, P, 1)(t, d^{-1}Q, d^{-1})(r, P, 1)(s, Q, d)(r, P, 1) \\
&= (r, P, 1)(s, Q \cup dP_r^1 \cup d(d^{-1}Q)_t^1 \cup P_r^1 \cup Q_s^1, d)(r, P, 1) \\
&= (r, P, 1)(s, Q \cup dP_r^1 \cup \{(d, t)\} \cup P_r^1, d)(r, P, 1) \\
&= (r, P, 1)(s, Q, d)(r, P, 1).
\end{aligned}$$

Similarly, we could show that

$$\begin{aligned} & ((r, P, 1)(t, d^{-1}Q, d^{-1})(r, P, 1))((r, P, 1)(s, Q, d)(r, P, 1))((r, P, 1)(t, d^{-1}Q, d^{-1})(r, P, 1)) \\ &= ((r, P, 1)(t, d^{-1}Q, d^{-1})(r, P, 1)). \end{aligned}$$

Thus we see that every element of $(r, P, 1)\text{Reg}(\mathcal{M}(S; \Sigma))(r, P, 1)$ has an inverse. Since all the elements of this subsemigroup have the same root, its idempotents must commute by (c). Moreover $(r, P, 1)\text{Reg}(\mathcal{M}(S; \Sigma))(r, P, 1)$ has identity element $(r, P, 1)$. Therefore it follows that $(r, P, 1)\text{Reg}(\mathcal{M}(S; \Sigma))(r, P, 1)$ is an inverse submonoid, which implies that $\text{Reg}(\mathcal{M}(S; \Sigma))$ is locally an inverse monoid. \blacksquare

The following example illustrates the properties described in Proposition 4.2.1.

Example: We investigate the graph expansion of the free group $S = gp\langle x | \emptyset \rangle$. We generate S as a semigroup by the set $\{a, b\}$ with map $f : \{a, b\} \rightarrow S$ defined by $a \mapsto x$, $b \mapsto x^{-1}$. This yields the semigroup system $(S, \{a, b\}, f)$. The Cayley digraph $\text{Cay}(S; \{a, b\})$ is shown in Figure 4.3. We describe the idempotent, regular, inverse, and indecomposable elements of $\mathcal{M}(S; \{a, b\})$ in Table 4.1 on page 57.

The Path Expansion for Semigroup Systems of Groups

The next result is the analog of Proposition 4.2.1 for semigroup path expansions. When there is no difference between the result or proof for the path expansion and the corresponding result or proof for the graph expansion, we indicate this in the

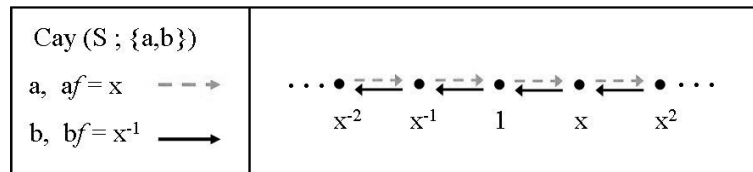


Figure 4.3: The Cayley digraph $\text{Cay}_{sgp}(S; \{a, b\})$ of the semigroup system $(S, \{a, b\}, a \mapsto x, b \mapsto x^{-1})$, where S is a free monogenic group.

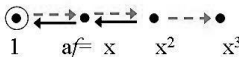
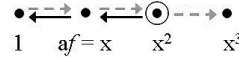
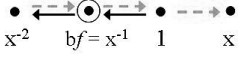
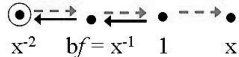
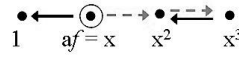
Special elements	Description	Example
Idempotents	Have form $(r, P, 1)$ where $P = P_r^1$; i.e. $(1, r) \in E(P)$.	$(a, A, 1)$ 
Regular elements	Have form (r, P, x^n) where $n \in \mathbb{Z}$, $P = P_r^1$ and the $x^n \rightarrow 1$ path is in P .	(a, A, x^2)  (b, B, x^{-1}) 
Inverses	Let (r, P, x^n) be regular. Its inverse(s) have the form $(s, x^{-n}P, x^{-n})$ where $s \in \{a, b\}$, $(x^n, s) \in E(P)$, and $[x^n(sf) \rightarrow 1] \subseteq P$.	(a, A, x^2) shown above and (b, B, x) 
Indecomposable elements	Have form (a, P, x) with $(1, a), (x^2, b) \notin E(P)$ or have form (b, P, x^{-1}) with $(1, b), (x^{-2}, a) \notin E(P)$.	(a, D, x) 

Table 4.1: Descriptions and examples of special elements of the graph expansion $\mathcal{M}(S; \{a, b\})$ where S is a free group. ■

proposition with the comment “no change” and omit the proof.

Proposition 4.2.2. *Let (S, Σ, f) be a semigroup system of a group S . The following are true about $\text{Path}(S; \Sigma)$:*

- (a') *an element (r, P, c) is idempotent if and only if $c = 1$ and $P = P_r^1$ (no change);*
- (b') *$E(\text{Path}(S; \Sigma))$ is a subsemigroup;*
- (c') *let $(r, P, 1), (s, Q, 1) \in E(\text{Path}(S; \Sigma))$; then $(r, P, 1)$ and $(s, Q, 1)$ commute if and only if $r = s$ (no change);*
- (d') *fix $r \in \Sigma$; the set $\{(r, P, 1) \mid (r, P, 1) \in E(\text{Path}(S; \Sigma))\}$ is a semilattice;*
- (e') *an element (r, P, c) is a regular element of $\text{Path}(S; \Sigma)$ if and only if $P = P_r^1$ and P is strongly connected;*
- (f') *if (r, P, c) is regular, then (s, Q, d) is an inverse if and only if $d = c^{-1}$, $Q = c^{-1}P$, and $(c, s) \in E(P)$;*
- (g') *$\text{Reg}(\text{Path}(S; \Sigma))$ is a subsemigroup; if $|\Sigma| = 1$, then $\text{Reg}(\text{Path}(S; \Sigma)) \cong \text{Reg}(\mathcal{M}(S; \Sigma)) \cong S$; if $|\Sigma| \geq 2$, then $\text{Reg}(\text{Path}(S; \Sigma))$ is not an inverse semigroup, but is locally an inverse monoid.*

Proof: We start by showing (b'). Let $(r, P, 1), (s, Q, 1) \in E(\text{Path}(S; \Sigma))$. Consider the product $(r, P, 1)(s, Q, 1)$. From Proposition 3.3.4(a), it is an element of $\text{Path}(S; \Sigma)$ and from Proposition 4.2.1(a), it is also idempotent. Thus $(r, P, 1)(s, Q, 1) \in E(\text{Path}(S; \Sigma))$, whereupon $E(\text{Path}(S; \Sigma))$ is a subsemigroup.

Part (d') follows from (c'). Turning to (e'), let $(r, P, c) \in \text{Path}(S; \Sigma)$. Suppose (r, P, c) is regular and let $x, y \in V(P)$. Proposition 4.2.1(e), says that $P = P_r^1$ and that there is a $c \rightarrow 1$ path in P . As P is 1-rooted, we know there exist $1 \rightarrow x$ and $1 \rightarrow y$ paths. From the definition of path expansion, there also exist $x \rightarrow c$ and $y \rightarrow c$ paths. Using these, we can create paths from x to y and vice-versa. Thus P is strongly connected. Conversely, if $P = P_r^1$ and P is strongly connected, then $1 \in V(P)$ and there is a $c \rightarrow 1$ path in P . Hence by Proposition 4.2.1(e), the element (r, P, c) is regular.

Next we turn to prove (f'). We assume that (r, P, c) is regular. If (s, Q, d) is an inverse of (r, P, c) in $\text{Path}(S; \Sigma)$, then it is in $\mathcal{M}(S; \Sigma)$ as well. Proposition 4.2.1(f) says that $d = c^{-1}$, $Q = c^{-1}P$, and $(c, s) \in E(P)$. Conversely, assume (r, P, c) is regular and suppose $d = c^{-1}$, $Q = c^{-1}P$, and $(c, s) \in E(P)$. Proposition 4.2.1(f) says that (s, Q, d) is an inverse of (r, P, c) in $\mathcal{M}(S; \Sigma)$. From (d'), we know that P is strongly connected. Thus Q is also strongly connected, from which it follows that $(s, Q, d) \in \text{Path}(S; \Sigma)$.

Lastly, we wish to show (g'). Suppose $(r, P, c), (s, Q, d) \in \text{Reg}(\text{Path}(S; \Sigma))$. Then from (e'), we know that $1 \in V(Q)$ and that P and Q are strongly connected. Hence, $P \cup cQ_s^1$ is also strongly connected. Since $P = P_r^1$, it follows that $P \cup cQ_s^1 = (P \cup cQ_s^1)_r^1$. Appealing again to (e'), we have that $(r, P, c)(s, Q, d)$ is a regular element of $\mathcal{M}(S; \Sigma)$. Since $\text{Path}(S; \Sigma)$ is closed under the operation, $(r, P, c)(s, Q, d)$ is also in $\text{Path}(S; \Sigma)$ and hence in $\text{Reg}(\text{Path}(S; \Sigma))$. We conclude that $\text{Reg}(\text{Path}(S; \Sigma))$ is a subsemigroup.

For the case when $|\Sigma| = 1$, the same type of argument as given in Proposition 4.2.1(g) shows that $\text{Reg}(\text{Path}(S; \Sigma)) \cong \text{Reg}(\text{Path}(S; \Sigma)) \cong S$. For the case when $\Sigma \geq 2$, the argument from Proposition 4.2.1(g) also applies, with a few slight modifications. Namely, we use $E(\text{Path}(S; \Sigma))$ and $\text{Reg}(\text{Path}(S; \Sigma))$ instead of $E(\mathcal{M}(S; \Sigma))$ and $\text{Reg}(\mathcal{M}(S; \Sigma))$ and thus make use of the result that $\text{Reg}(\text{Path}(S; \Sigma))$ is a subsemigroup. ■

Example: We return to the example of the free group $S = gp\langle x | \emptyset \rangle$ with the semigroup system $(S, \{a, b\}, a \mapsto x, b \mapsto x^{-1})$. This is the same semigroup system used in the example at the end of Section 4.2. For the path expansion, we can provide further description of idempotent, regular, and indecomposable elements. Namely, in light of Proposition 4.2.2(e'), all idempotent and regular elements in the path expansion have strongly connected subdigraphs. In the free group with one generator, the strongly connected digraphs are all graphs. This information is incorporated into Table 4.2. ■

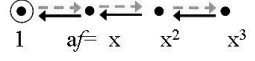
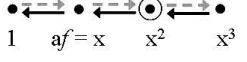
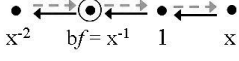
Special elements	Description	Example
Idempotents	Have form $(r, P, 1)$ where P is a graph.	$(a, A, 1)$ 
Regular elements	Have form (r, P, x^n) where $n \in \mathbb{Z}$, $1 \in V(P)$, and P is a graph.	(a, A, x^2)  (b, B, x^{-1}) 
Indecomposable elements	There are two: (a, \bullet_a, x) and (b, \bullet_b, x^{-1}) .	

Table 4.2: Descriptions and examples of special elements of the path expansion $\text{Path}(S; \{a, b\})$ where S is a free group.

4.3 Left-Zero Semigroups

Let (L, Σ, f) be a semigroup system of a left-zero semigroup L . The Cayley digraph $\text{Cay}(L; \Sigma)$ has $|L|$ components and each component contains one vertex with $|\Sigma|$ loops, each loop labeled by a different element of Σ . For example, the left-zero semigroup $L = \langle x, y \mid xy = x, yx = y \rangle$ generated by $\Sigma = \{r, s, t\}$ under the map $r \mapsto x$, $s \mapsto x$, and $t \mapsto y$ is shown in Figure 4.4.

As we see in Figure 4.4, the structure of the left-zero semigroup forces all elements of $\mathcal{M}(L; \Sigma)$ to have the form (r, P, rf) . Additionally, all rooted digraphs are determined by their sets of edge labels. Thus if we know P is a rooted digraph and we are given the edge label set $\Sigma(P)$ and its root c , it is easy to reconstruct P . This enables us to give an alternative description of graph expansions of left-zero semigroups:

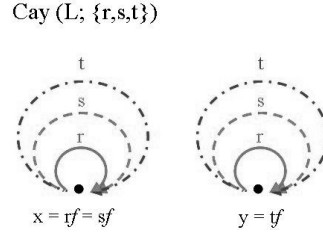


Figure 4.4: The Cayley digraph of a left-zero semigroup of order two generated by a 3 element set.

Lemma 4.3.1. *Let (L, Σ, f) be a semigroup system of a left-zero semigroup L . Then $\mathcal{M}(L; \Sigma)$ is isomorphic to the semigroup with elements from $\Sigma \times \mathcal{P}(\Sigma)$ and binary operation:*

$$(a, A)(b, B) = (a, A \cup B \cup \{b\}).$$

Proof: Use the map $\mathcal{M}(L; \Sigma) \rightarrow L \times \mathcal{P}(\Sigma)$ given by $(r, P, rf) \mapsto (r, \Sigma(P))$. It is easy to check that this map is an injective and surjective homomorphism, and thus an isomorphism. ■

Using this alternative description, it is clear that the order of Σ , not the order of L , most influences the structure of $\mathcal{M}(L; \Sigma)$. We now state many of the basic properties of $\mathcal{M}(L; \Sigma)$, giving elements in the form $(r, \Sigma(P))$.

Proposition 4.3.2. *Let (L, Σ, f) be a semigroup system of a left-zero semigroup L . The following are true about $\mathcal{M}(L; \Sigma)$:*

- (a) *an element $(r, \Sigma(P))$ is idempotent if and only if $r \in \Sigma(P)$;*
- (b) *$E(\mathcal{M}(L; \Sigma))$ is a subsemigroup;*
- (c) *elements $(r, \Sigma(P))$ and $(s, \Sigma(Q))$ commute if and only if $r = s$;*
- (d) *an element $(r, \Sigma(P))$ is regular if and only if it is idempotent;*
- (e) *if $(r, \Sigma(P))$ and $(s, \Sigma(Q))$ are regular, then they are inverses of each other if and only if $\Sigma(P) = \Sigma(Q)$;*
- (f) *if $|\Sigma| = 1$, then $E(\mathcal{M}(L; \Sigma))$ is the trivial group; if $|\Sigma| > 1$, then $E(\mathcal{M}(L; \Sigma))$ is not commutative;*

(g) $\mathcal{M}(L; \Sigma)$ is locally a semilattice; the local subsemigroup obtained with idempotent $(r, \Sigma(P))$ is $\{(r, T) \mid \Sigma(P) \subseteq T\}$; it is isomorphic to the semigroup of all subsets of Σ containing $\Sigma(P)$ with the operation union.

Proof: Properties (a) through (e) are easily obtained using the description of multiplication given in Lemma 4.3.1. For (f), if $|\Sigma| = 1$, then $E(\mathcal{M}(L; \Sigma))$ contains precisely one idempotent and is by default trivial. If $|\Sigma| > 1$, then we can use (a) and (c) to find two idempotents that do not commute.

We wish to show that $\mathcal{M}(L; \Sigma)$ is locally a semilattice. Let $(r, \Sigma(P))$ be an idempotent and let $(s, \Sigma(Q)) \in \mathcal{M}(L; \Sigma)$. By (a), we know $r \in \Sigma(P)$ and hence the product

$$(r, \Sigma(P))(s, \Sigma(Q))(r, \Sigma(P)) = (r, \Sigma(P) \cup \Sigma(Q) \cup \{s\}) \quad (4.3.1)$$

is also idempotent. By (c), all idempotents with first entry r commute. Thus the local subsemigroup at $(r, \Sigma(P))$ is a semilattice. The map $(r, T) \mapsto T$ sends this local subsemigroup to the semigroup of all subsets of Σ containing $\Sigma(P)$. It is clearly injective, surjective, and preserves the operation, and is thus an isomorphism. ■

The Path Expansion for Left-Zero Semigroups

Since all rooted digraphs in $\text{Cay}(L; \Sigma)$ can be traversed by paths, we have the following result:

Proposition 4.3.3. *Let (L, Σ, f) be a semigroup system of a left-zero semigroup L . Then $\text{Path}(L; \Sigma) \cong \mathcal{M}(L; \Sigma)$.*

4.4 Right-Zero Semigroups

In the previous chapter we gave an example of the Cayley digraph of a right-zero semigroup with three elements generated by a three element set (see Figure 3.1 on

page 26). As indicated in Figure 3.1, the Cayley digraph of a right-zero semigroup system (R, Σ, f) has the following form: for every element $a \in R$ and every $r \in \Sigma$, the digraph $\text{Cay}(R; \Sigma)$ contains an r -labeled edge from a to rf . The Cayley digraph also has the useful property that the left action of S on $\text{Cay}(R; \Sigma)$ (corresponding to left translation by elements of S) is trivial. In other words, if $c \in S$ and $P \subseteq \text{Cay}(R; \Sigma)$, then $cP = P$. This enables us to simplify the product for graph expansion elements.

Lemma 4.4.1. *Let (R, Σ, f) be a semigroup system of a right-zero semigroup R . Then $(r, P, c)(s, Q, d) = (r, P \cup Q \cup \{(c, s)\}, d)$.*

Proof: We derive that:

$$\begin{aligned}
 (r, P, c)(s, Q, d) &= (r, P \cup c(Q_s^1), cd) \\
 &= (r, P \cup cQ \cup \{c(1, s)\}, d) \\
 &= (r, P \cup Q \cup \{(c, s)\}, d). \quad \blacksquare
 \end{aligned}$$

This simplification will be useful as we describe many of the properties of right-zero semigroup graph expansions.

Proposition 4.4.2. *Let (R, Σ, f) be a semigroup system of a right-zero semigroup R . The following are true about $\mathcal{M}(R; \Sigma)$:*

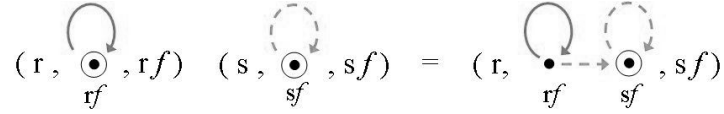
- (a) *an element (r, P, c) is idempotent if and only if $(c, r) \in E(P)$;*
- (b) *$E(\mathcal{M}(R; \Sigma))$ is a subsemigroup if and only if $|R| = 1$;*
- (c) *elements (r, P, c) and (s, Q, d) commute if and only if $s = r$ and $c = d$;*
- (d) *an element (r, P, c) is regular if and only if there is a non-empty $c \longrightarrow rf$ path in P ;*
- (e) *$\text{Reg}(\mathcal{M}(R; \Sigma))$ is a subsemigroup if and only if $|R| = 1$; moreover, if $|R| = 1$, then $\text{Reg}(\mathcal{M}(R; \Sigma)) = E(\mathcal{M}(R; \Sigma))$;*

(f) if (r, P, c) is regular, then (s, Q, d) is an inverse if and only if $P = Q$ and $(c, s), (d, r) \in E(P)$;

(g) $\mathcal{M}(R; \Sigma)$ is locally a semilattice.

Proof: To prove part (a), suppose (r, P, c) is idempotent. Then from Theorem 3.2.1 (b), $cP_r^1 \in P$, which gives that $(c, r) \in E(P)$. Conversely, if $(c, r) \in E(P)$, then $cP_r^1 = cP \cup \{(c, r)\} = P$. Again appealing to Theorem 3.2.1 (b), (r, P, c) is idempotent.

Moving on to part (b), suppose $E(\mathcal{M}(R; \Sigma))$ is a subsemigroup. Let $r, s \in \Sigma$. We wish to show that $rf = sf$. Let P be the digraph consisting of the r -labeled loop at rf . Similarly, let Q be the digraph consisting of the s -labeled loop at sf . By part (a), both (r, P, rf) and (s, Q, sf) are idempotent elements of $\mathcal{M}(R; \Sigma)$. Moreover, the element $(r, P, rf)(s, Q, sf)$, which we show below, is idempotent.



Thus from part (a), the digraph $P \cup (rf)Q_s^1$ contains the edge (sf, r) . This implies that $rf = sf$. Hence $|R| = 1$. For the converse, assume $|R| = 1$. Let (r, P, c) and (s, Q, c) be idempotents. From (a), we know that $(c, r) \in E(P)$. Hence $(c, r) \in E(P \cup Q)$, whereupon we see that the product is idempotent.

In order to show (c), suppose elements (r, P, c) and (s, Q, d) commute. Then $(r, P \cup cQ_s^1, d) = (s, Q \cup dP_r^1, c)$. Immediately we see that $r = s$ and $c = d$. To prove the converse, assume that $r = s$ and $c = d$. Let $(r, P, c), (r, Q, c) \in \mathcal{M}(R; \Sigma)$. Recall that left-translation corresponds to the identity map for subdigraphs, i.e. that

$cP = P$ and $cQ = Q$. Therefore,

$$\begin{aligned}
(r, P, c)(r, Q, c) &= (r, P \cup cQ_r^1, c) \\
&= (r, cP \cup Q \cup \{(c, r)\}, c) \\
&= (r, Q \cup cP_r^1, c) \\
&= (r, Q, c)(r, P, c).
\end{aligned}$$

We now wish to prove (d). Suppose (r, P, c) is regular. Let (s, Q, d) be an inverse. Then it follows that

$$(r, P, c) = (r, P, c)(s, Q, d)(r, P, c) = (r, P \cup Q \cup \{(c, s), (d, r)\}, c).$$

The subdigraph Q , by virtue of being sf -rooted, contains an $sf \rightarrow d$ path. Affixing the edges (c, s) and (d, r) to this path produces the desired non-empty $c \rightarrow rf$ path in P . For the converse, suppose that (r, P, c) is such that P contains a non-empty $c \rightarrow rf$ path. Thus, there exists some vertex $x \in V(P)$ such that $(x, r) \in E(P)$. From the properties of right zero semigroups, $xc = c$ and $xP_r^1 = P$. Moreover, if we combine the $c \rightarrow rf$ path with a $rf \rightarrow x$ path, we obtain a $c \rightarrow x$ path in P . Using Theorem 3.2.1(c), we conclude that (r, P, c) is regular.

We turn to proving (e). The same argument as used in part (b) shows that if $\text{Reg}(\mathcal{M}(R; \Sigma))$ is a subsemigroup, then $|R| = 1$. Conversely, if $|R| = 1$, then (a) and (d) imply that all regular elements are idempotent, i.e. we obtain that $\text{Reg}(\mathcal{M}(R; \Sigma)) = E(\mathcal{M}(R; \Sigma))$. Thus (b) implies that $\text{Reg}(\mathcal{M}(R; \Sigma))$ is a subsemigroup.

In order to show (f), suppose that (r, P, c) and (s, Q, d) are inverses. Then

$$(r, P, c) = (r, P, c)(s, Q, d)(r, P, c) = (r, P \cup Q \cup \{(c, s), (d, r)\}, c).$$

We see immediately that $Q \subseteq P$ and $(c, s), (d, r) \in E(P)$. The corresponding equation

with (s, Q, d) on the left hand side shows that $P \subseteq Q$. Hence $P = Q$. On the other hand, if we assume that $P = Q$ and $(c, s), (d, r) \in E(P)$, then the converse is evident upon examining the relevant products.

Finally, we turn to the proof of (g). Assume (r, P, c) is idempotent and let $(s, Q, d) \in \mathcal{M}(S; \Sigma)$. Consider the element

$$(r, P, c)(s, Q, d)(r, P, c) = (r, P \cup Q \cup \{(c, s), (d, r)\}, c).$$

Since $(c, r) \in E(P) \subseteq E(P \cup Q \cup \{(c, s), (d, r)\})$, we know by (a) that the element $(r, P, c)(s, Q, d)(r, P, c)$ is idempotent. Appealing to (c), all elements of $(r, P, c)\mathcal{M}(S; \Sigma)(r, P, c)$ commute. Thus $\mathcal{M}(S; \Sigma)$ is locally a semilattice. ■

The Path Expansion for Right-Zero Semigroups

The properties of the graph expansion of right-zero semigroup systems carry over to the path expansion. We give these results in Proposition 4.4.3. As before, when there is no difference between the result or proof for the path expansion with that for the graph expansion in Proposition 4.4.2, we indicate this with the comment “no change” and omit the proof.

Proposition 4.4.3. *Let (R, Σ, f) be a semigroup system of a right-zero semigroup R . The following are true about $\text{Path}(R; \Sigma)$:*

- (a') *an element (r, P, c) is idempotent if and only if $(c, r) \in E(P)$ (no change);*
- (b') *$E(\text{Path}(R; \Sigma))$ is a subsemigroup if and only if $|R| = 1$;*
- (c') *elements (r, P, c) and (s, Q, d) commute if and only if $s = r$ and $c = d$ (no change);*
- (d') *an element (r, P, c) is regular if and only if P is strongly connected;*
- (e') *$\text{Reg}(\text{Path}(R; \Sigma))$ is a subsemigroup if and only if $|R| = 1$; moreover, if $|R| = 1$, then $\text{Reg}(\text{Path}(R; \Sigma)) = E(\text{Path}(R; \Sigma))$;*

- (f') if $(r, P, c) \in \text{Path}(S; \Sigma)$ is regular, then (s, Q, d) is an inverse if and only if $P = Q$ and $(c, s), (d, r) \in E(P)$;
- (g') $\text{Path}(R; \Sigma)$ is locally a semilattice.

Proof: We start with a proof of (b'). If $E(\text{Path}(R; \Sigma))$ is a subsemigroup, then the same argument as given for Proposition 4.4.2(b) shows that $|R| = 1$. Conversely, if $|R| = 1$, then R is also a left zero semigroup and by Proposition 4.3.3, we know that $\text{Path}(R; \Sigma) = \mathcal{M}(R; \Sigma)$. From Proposition 4.3.2(b), we have that $E(\text{Path}(R; \Sigma))$ is a subsemigroup.

In order to show (d'), we suppose (r, P, c) is regular. Let $x, y \in V(P)$. Due to the properties of the path expansion, the digraph P contains the following paths: $rf \longrightarrow x$, $rf \longrightarrow y$, $x \longrightarrow c$, and $y \longrightarrow c$. Moreover, from the regularity of (r, P, c) , we know there is an element (s, Q, d) such that

$$(r, P, c) = (r, P, c)(s, Q, d)(r, P, c) = (r, P \cup Q \cup \{(c, s), (d, r)\}, c).$$

The digraph Q contains a path $sf \longrightarrow d$. If we attach the edges (c, s) and (d, r) to this path, we see that P contains a $c \longrightarrow rf$ path. Combining this path with the earlier paths, we have paths from x to y and vice versa. Thus P is strongly connected.

We now wish to prove (e'). If $\text{Reg}(\text{Path}(R; \Sigma))$ is a subsemigroup, then we can again rely on the argument used to prove Proposition 4.4.2(b) to show that $|R| = 1$. If $|R| = 1$, then as we noted before, R is a left zero semigroup and hence we have that $\text{Path}(R; \Sigma) = \mathcal{M}(R; \Sigma)$. Thus applying Proposition 4.3.2(d) yields that $E(\text{Path}(R; \Sigma)) = \text{Reg}(\text{Path}(R; \Sigma))$. Then (b') implies that $\text{Reg}(\text{Path}(R; \Sigma))$ is a subsemigroup.

One direction of (f') follows from Proposition 4.4.2(f). For the converse, assume that (r, P, c) is regular and that $(c, s), (d, r) \in E(P)$. From Proposition 4.4.2(f), we know that (s, P, d) is an inverse of (r, P, c) in $\mathcal{M}(S; \Sigma)$. We wish to show that (s, P, d) is in the path expansion. Since $(r, P, c) \in \text{Path}(S; \Sigma)$, there is a path $rf \longrightarrow c$ in P that traverses every edge of P . Similarly, as $(s, P, d) \in \mathcal{M}(S; \Sigma)$ there is a $sf \longrightarrow d$ in

P . Combining these two paths via the edge (c, s) , which we assumed to be in $E(P)$, we obtain a path $sf \longrightarrow d$ which traverses every edge of P . Thus $(s, P, d) \in \text{Path}(S; \Sigma)$, as desired.

Finally, if we replace $E(\mathcal{M}(R; \Sigma))$ by $E(\text{Path}(R; \Sigma))$ in the proof of Proposition 4.4.2(g), we obtain a proof of (g'). ■

4.5 Semigroup Direct Products with Left-Zero Factor, in Particular Rectangular Bands

In this section we investigate semigroup direct products of the form $L \times S$, where L is a left-zero semigroup and S is any semigroup. The direct product is equipped with homomorphisms onto its factors, which we denote by π_L and π_S respectively. Consider a system $(L \times S, \Sigma, f)$. We can use this system to form a system for S , namely $(S, \Sigma, f \circ \pi_S)$. (We can just as easily form one for L , but it will not be needed here.) Note that the map $\pi_S : L \times S \rightarrow S$ is a Σ -preserving semigroup system homomorphism. The map π_S induces two additional maps, one between subdigraphs of Cayley digraphs and one between graph expansions.

$$\begin{aligned} \widehat{\pi_S} : \mathcal{P}(\text{Cay}(L \times S; \Sigma)) &\rightarrow \mathcal{P}(\text{Cay}(S; \Sigma)) && \text{the maps between subdigraphs are determined} \\ \text{Vertices: } \pi_S : L \times S &\rightarrow S; \\ \text{Edges: } (x, r) &\text{ to } (x\pi_S, r); \end{aligned}$$

$$\check{\pi_S} : \mathcal{M}(L \times S; \Sigma) \rightarrow \mathcal{M}(S; \Sigma) \qquad (r, P, c) \mapsto (r, P\widehat{\pi_S}, c\pi_S).$$

For intuition about $\widehat{\pi_S}$, consider that the Cayley digraph $\text{Cay}(L \times S; \Sigma)$ consists of $|L|$ disjoint copies of $\text{Cay}(S; \Sigma)$, each copy indexed by an element of L and isomorphic, as a Σ -labeled graph, to $\text{Cay}(S; \Sigma)$. The map $\widehat{\pi_S}$ projects the copies onto their corresponding parts in $\text{Cay}(S; \Sigma)$. When restricted to single components, this map is

injective. This leads to the following Lemma.

Lemma 4.5.1. *Let $(L \times S, \Sigma, f)$ be a semigroup system where L is a left-zero semigroup and S is any semigroup. If $r \in \Sigma$ and P is an $rf = (\ell_r, x_r)$ -rooted digraph, then every vertex in P has first coordinate ℓ_r .*

In light of Lemma 4.5.1, we will express all elements in the form $(r, P, (\ell_r, c))$. From this lemma, we see that $\widehat{\pi_S}$ embeds rooted-digraphs of $\text{Cay}(L \times S; \Sigma)$ in $\text{Cay}(S; \Sigma)$. We are thus able to show the following:

Proposition 4.5.2. *Let $(L \times S, \Sigma, f)$ be a semigroup system where L is a left-zero semigroup and S is any semigroup. If we form the semigroup system $(S, \Sigma, f \circ \pi_S)$, then it follows that $\mathcal{M}(L \times S; \Sigma) \cong \mathcal{M}(S; \Sigma)$ with an isomorphism given by $\check{\pi}_S$ as defined above.*

Proof: We wish to show that $\check{\pi}_S : \mathcal{M}(L \times S; \Sigma) \rightarrow \mathcal{M}(S; \Sigma)$ is a bijective homomorphism. In Lemma 3.2.5(c), we showed that $\check{\pi}_S$ is a semigroup homomorphism. We focus on showing that it is injective. Suppose $(r, P, (\ell_r, c)), (s, Q, (\ell_s, d)) \in \mathcal{M}(L \times S; \Sigma)$ are such that $(r, P, (\ell_r, c))\check{\pi}_S = (s, Q, (\ell_s, d))\check{\pi}_S$. This implies that $(r, P\widehat{\pi_S}, c) = (s, Q\widehat{\pi_S}, d)$, from which we see that $r = s$, $P\widehat{\pi_S} = Q\widehat{\pi_S}$, and $c = d$. We now prove that $P = Q$. If $E(P) = \emptyset$, then we have $E(Q) = \emptyset$, since $E(Q\widehat{\pi_S}) = E(P\widehat{\pi_S}) = \emptyset$. In this case, P and Q both consist of the vertex at rf and are equal. If $E(P) \neq \emptyset$, then the rootedness of P and Q means that we can show that $P = Q$ by showing $E(P) = E(Q)$. Let $((\ell_r, y), t) \in E(P)$. Then $(y, t) \in E(P\widehat{\pi_S}) = E(Q\widehat{\pi_S})$. This implies that there exists some $((\ell_k, y), t) \in E(Q)$. Since (ℓ_k, y) is a vertex in a graph rooted at $rf = (\ell_r, x_r)$, by Lemma 4.5.1, $\ell_k = \ell_r$. Hence $((\ell_r, y), t) \in E(Q)$, whereupon $E(P) \subseteq E(Q)$. The reverse inclusion can be shown similarly. We conclude that $E(P) = E(Q)$ and thus $P = Q$. This shows that the map $\check{\pi}_S$ is injective.

Suppose $(r, P, c) \in \mathcal{M}(S; \Sigma)$. Let P' be the copy of P rooted at (ℓ_r, x_r) and let (ℓ_r, c') be the vertex corresponding to c . Then $(r, P', (\ell_r, c'))\check{\pi}_S = (r, P, c)$. We conclude that $\check{\pi}_S$ is surjective and hence a bijection. ■

If we re-examine left-zero semigroup systems in light of Proposition 4.5.2, this result implies that the graph expansion of a Σ -generated left-zero semigroup system is isomorphic to the graph expansion of the Σ -generated trivial group. By using Proposition 4.5.2, we can derive much of Proposition 4.3.2 from Proposition 4.2.1: namely $4.2.1(a) \rightarrow 4.3.2(a)$, $4.2.1(b) \rightarrow 4.3.2(b)$, $4.2.1(c) \rightarrow 4.3.2(c)$, $4.2.1(e)$ and $4.3.2(a) \rightarrow 4.3.2(d)$, $4.2.1(f) \rightarrow 4.3.2(e)$, $4.2.1(g)$ and $4.3.2(f) \rightarrow 4.3.2(f)$.

Proposition 4.5.2 also indicates an important way in which the semigroup graph expansion differs from other graph expansions. Consider the group case: if we know an inverse semigroup is a graph expansion of some group system, then we know that the group involved is the maximal group image of the semigroup (see [17]). Similar relationships hold for right cancellative monoid, unipotent, and inverse semigroup graph expansions (see [7],[6], and [15]). In the semigroup case, knowing a semigroup is a graph expansion of some semigroup system does not allow us to determine the original semigroup. Thinking in terms of functors, Proposition 4.5.2 has the following implication:

Corollary 4.5.3. *The functor $\mathcal{M}(_, _)$ from the category SGP_Σ to the category SGP is not injective.*

We give an example of two non-isomorphic semigroups whose graph expansions are isomorphic:

Example: Let L be the left-zero semigroup with three elements,

$$L = \langle x_1, x_2, x_3 \mid x_i x_j = x_i \text{ for } i, j = 1, 2, 3 \rangle$$

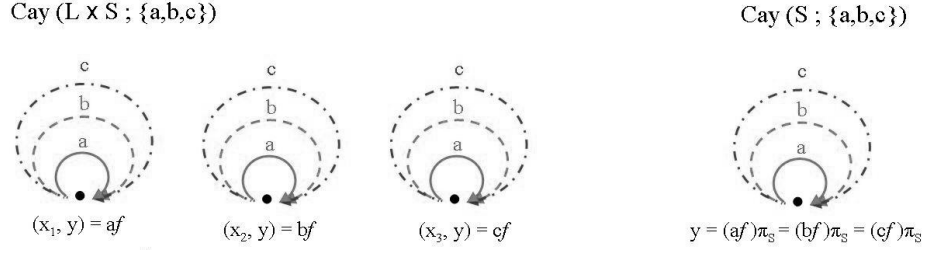
and S be the trivial group,

$$S = \langle y \mid y^2 = y \rangle.$$

Choose $\Sigma = \{a, b, c\}$ and use the map $f : \Sigma \rightarrow L \times S$, given by

$$\begin{aligned} a &\mapsto (x_1, y) \\ b &\mapsto (x_2, y) \\ c &\mapsto (x_3, y) \end{aligned}$$

The Cayley graphs $\text{Cay}(L \times S; \Sigma)$ and $\text{Cay}(S; \Sigma)$ are shown below:



The graph expansions $\mathcal{M}(L \times S; \Sigma)$ and $\mathcal{M}(S; \Sigma)$ have 24 elements. They are isomorphic to the set $\{a, b, c\} \times \mathcal{P}(\{a, b, c\})$ with the operation $(r, A)(s, B) = (r, A \cup B \cup \{s\})$.

Let $P \subseteq \text{Cay}(L \times S; \Sigma)$ be the subdigraph rooted at (x_1, y) containing exactly the b -labeled edge. Let $P' \subseteq \text{Cay}(S; \Sigma)$ be the subdigraph rooted at y containing exactly the b -labeled edge. Then the isomorphism referred to in Proposition 4.5.2, matches the element $(a, P, (x_1, y)) \in \mathcal{M}(L \times S; \Sigma)$ with $(a, P', y) \in \mathcal{M}(S; \Sigma)$. ■

Combining Proposition 4.4.2 (about right-zero semigroups) and Proposition 4.5.2 (about direct products with left-zero factor) enables us to describe many properties of the graph expansion of a rectangular band. Note that of the results below, (f) is the only one which differs, albeit in a minor way, from the corresponding result in Proposition 4.4.2.

Corollary 4.5.4. *Let $(L \times R, \Sigma, f)$ be a semigroup system of a rectangular band $L \times R$. Then the following are true about the semigroup graph expansion $\mathcal{M}(L \times R; \Sigma)$:*

- (a) *an element (r, P, c) is idempotent if and only if $(c, r) \in E(P)$;*
- (b) *$E(\mathcal{M}(L \times R; \Sigma))$ is a subsemigroup if and only if $|R| = 1$;*
- (c) *elements (r, P, c) and (s, Q, d) commute if and only if $s = r, c = d$;*
- (d) *an element (r, P, c) is regular if and only if there is a $c \longrightarrow rf$ path in P ;*
- (e) *$\text{Reg}(\mathcal{M}(R; \Sigma))$ is a subsemigroup if and only if $|R| = 1$; moreover, if $|R| = 1$, then $\text{Reg}(\mathcal{M}(R; \Sigma)) = E(\mathcal{M}(R; \Sigma))$;*
- (f) *if (r, P, c) is regular, then (s, Q, d) is an inverse if and only if $P\widehat{\pi_R} = Q\widehat{\pi_R}$ and $(c\pi_R, s), (d\pi_R, r) \in E(P\widehat{\pi_R})$;*

(g) $\mathcal{M}(L \times R; \Sigma)$ is locally a semilattice.

Similar results can be shown for the graph expansions of direct products of left-zero semigroups and groups, left-zero semigroups, semilattices, and so on.

The Path Expansion for Direct Products with Left-Zero Factor

We can replace the graph expansion by the path expansion in Proposition 4.5.2 and Corollary 4.5.3. Aside from replacing $\mathcal{M}(L \times S; \Sigma)$ by $\text{Path}(L \times S; \Sigma)$ and $\mathcal{M}(S; \Sigma)$ by $\text{Path}(S; \Sigma)$, the proofs remain unchanged.

Proposition 4.5.5. *Let $(L \times S, \Sigma, f)$ be a semigroup system where L is a left-zero semigroup and S is any semigroup. If we form the semigroup system $(S, \Sigma, f \circ \pi_S)$, then it follows that $\text{Path}(L \times S; \Sigma) \cong \text{Path}(S; \Sigma)$.*

Corollary 4.5.6. *The functor $\text{Path}(_, _)$ from the category SGP_Σ to itself is not injective.*

We can also describe the properties of a path expansion of a rectangular band. These results follow from Proposition 4.4.3 and 4.5.5.

Corollary 4.5.7. *Let $(L \times R, \Sigma, f)$ be a semigroup system of a rectangular band $L \times R$. Then the following are true about the semigroup graph expansion $\text{Path}(L \times R; \Sigma)$:*

- (a') *an element (r, P, c) is idempotent if and only if $(c, r) \in E(P)$ (no change);*
- (b') *$E(\text{Path}(L \times R; \Sigma))$ is a subsemigroup if and only if $|R| = 1$;*
- (c') *elements (r, P, c) and (s, Q, d) commute if and only if $s = r, c = d$;*
- (d') *an element (r, P, c) is regular if and only if P is strongly connected;*
- (e') *$\text{Reg}(\text{Path}(R; \Sigma))$ is a subsemigroup if and only if $|R| = 1$; moreover, if $|R| = 1$, then $\text{Reg}(\text{Path}(R; \Sigma)) = E(\text{Path}(R; \Sigma))$;*

- (f') if (r, P, c) is regular, then (s, Q, d) is an inverse if and only if $P\widehat{\pi_R} = Q\widehat{\pi_R}$
and $(c\pi_R, s), (d\pi_R, r) \in E(P\widehat{\pi_R})$;
- (g') $\text{Path}(L \times R; \Sigma)$ is locally a semilattice.

4.6 Semilattices

We now explore graph expansions of semilattices. In contrast to groups and right-zero semigroups, in the Cayley digraphs of semilattices there are no cycles passing through two or more vertices. In other words, if (S, Σ, f) is a semigroup system of a semilattice S , and a and b are distinct elements of S , then exactly one of the following holds:

1. there is a path from a to b but no path from b to a ;
2. there is a path from b to a but no path from a to b ;
3. there is no path from a to b nor from b to a .

Cayley digraphs of semilattice systems also contain many loops. For example, for every r -labeled edge that lies on a path ending at the vertex a , there is an r -labeled edge at a . An example of a Cayley digraph of a free semilattice with three generators is shown in Figure 4.5.

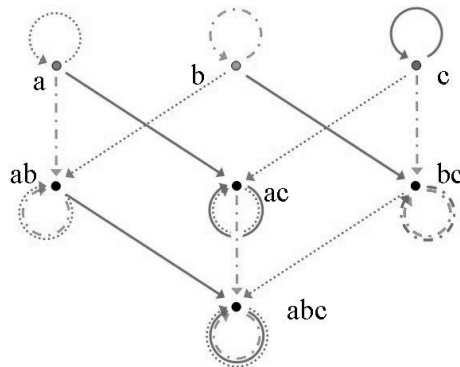


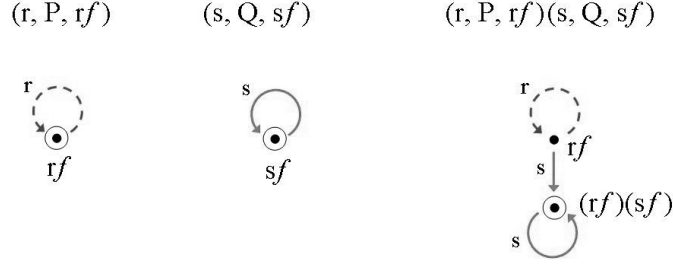
Figure 4.5: A free semilattice with three distinct generators.

In this section, the *natural order* for inverse semigroups will be useful: if S is an inverse semigroup and $a, b \in S$, then $a \leq b$ if and only if there exists some $c \in E(S)$ such that $a = bc$. We note that for semilattices, $S = E(S)$. Also, in the following proposition we use the subdigraph $P_c^{\uparrow\uparrow}$. Recall that if $P \subset \Gamma$ and $c \in V(P)$, then $P_c^{\uparrow\uparrow}$ is the maximal subdigraph of P which is accessible from c along paths in Γ , with isolated vertices removed.

Proposition 4.6.1. *Let (S, Σ, f) be a semigroup system of a semilattice S . The following are true about $\mathcal{M}(S; \Sigma)$:*

- (a) (r, P, c) is an idempotent if and only if $cP_r^1 \subseteq P$;
- (b) $E(\mathcal{M}(S; \Sigma))$ is a subsemigroup if and only if $|S| = 1$;
- (c) if elements (r, P, c) and (s, Q, d) commute, then $r = s$ and the following are true about P and Q
 - 1. if $c \notin V(Q)$, then $c < d$ and $d \in V(P)$;
 - 2. if $d \notin V(P)$, then $d < c$ and $c \in V(Q)$;
 - 3. if $c \in V(Q)$, then $Q \setminus Q_c^{\uparrow\uparrow} \subseteq P$;
 - 4. if $d \in V(P)$, then $P \setminus P_d^{\uparrow\uparrow} \subseteq Q$;
- (d) an element (r, P, c) is regular if and only if it is idempotent;
- (e) if (r, P, c) and (s, Q, d) are regular, then (r, P, c) has (s, Q, d) as an inverse if and only if $c = d$ and $P_c^{\uparrow} = Q_c^{\uparrow}$.

Proof: The description of idempotents given in part (a) is the same as for the general case in Theorem 3.2.1(b). We thus start by showing (b). Suppose $|S| = 1$. Then S is the trivial group and part (b) follows from Proposition 4.2.1(b). We show the converse with a contrapositive approach. Suppose $|S| > 1$. Then we can choose some $(r, P, rf), (s, Q, sf)$ with $rf \neq sf$ as shown below:



The product $(r, P, rf)(s, Q, sf) = (r, P \cup (rf)Q_s^1, (rs)f)$ is not idempotent, because it does not contain an r -labeled loop at vertex $(rs)f$. Hence for $|S| > 1$, $E(\mathcal{M}(S; \Sigma))$ is not a subsemigroup.

Turning to (c), suppose that the elements (r, P, c) and (r, Q, d) commute, i.e. that $(r, P \cup cQ_s^1, cd) = (s, Q \cup dP_r^1, cd)$. We see immediately that $r = s$ (we could also have used Theorem 3.2.1 (d)). We thus replace s by r for the remainder of the proof. We now show 1., noting that 2. can be similarly shown. Suppose $c \notin V(Q)$. Thus $c \neq d$. Then, as $c \in V(P) \subseteq V(P \cup cQ_s^1) = V(Q \cup dP_r^1)$, we have that $c \in V(dP_r^1)$. It follows that $c < d$, whereupon $d \notin V(cQ_s^1)$. However, as $d \in V(Q) \subseteq V(Q \cup dP_r^1) = V(P \cup cQ_s^1)$, we see that $d \in V(P)$.

Next we show 3., again noting that 4. can be similarly shown. Suppose $c \in V(Q)$. Then $Q_c^{\uparrow\uparrow}$ is defined. Note that $(Q \setminus Q_c^{\uparrow\uparrow}) \cap cQ_s^1 = \emptyset$ since every vertex in cQ_s^1 is accessible from c and every vertex in $Q \setminus Q_c^{\uparrow\uparrow}$ is not accessible from c . However as

$$Q \setminus Q_c^{\uparrow\uparrow} \subseteq Q \subseteq Q \cup dP_r^1 = P \cup cQ_s^1,$$

it follows that $Q \setminus Q_c^{\uparrow\uparrow} \subseteq P$.

We now proceed to (d). Assume an element (r, P, c) is regular. Then it has an inverse (s, Q, d) satisfying the equation

$$(r, P, c) = (r, P \cup cQ_s^1 \cup cdP_r^1, cd). \quad (4.6.1)$$

We observe that $c = cd$. Hence, $cP_r^1 \subseteq P$, whereupon from part (a), (r, P, c) is idempotent. Since idempotent elements are always regular, this proves (d).

Finally, in order to establish part (e), suppose (r, P, c) and (s, Q, d) are regular and that (s, Q, d) is an inverse for (r, P, c) . Then $(r, P, c) = (r, P \cup cQ_s^1 \cup cdP_r^1, cd)$, whereupon $c = cd$. We could similarly show that $d = dc$. Since S is a semilattice, $cd = dc$ and we have that $c = d$. From (d) we know that (s, Q, c) is idempotent. Thus from (a), $cQ_s^1 \subseteq Q$. This implies that $Q_c^\uparrow = cQ_s^1$. By Equation 4.6.1, $cQ_s^1 \subseteq P$. However, since Q_c^\uparrow is rooted at c , $Q_c^\uparrow \subseteq P_c^\uparrow$. We can derive the reverse containment from the equation $(s, Q, d) = (s, Q, d)(r, P, c)(s, Q, d)$. Thus $P_c^\uparrow = Q_c^\uparrow$.

Conversely, suppose (r, P, c) and (s, Q, c) are regular and that $P_c^\uparrow = Q_c^\uparrow$. Because (s, Q, c) is idempotent, we know that $cQ_s^1 \subseteq Q$ and hence that $cQ_s^1 \subseteq Q_c^\uparrow = P_c^\uparrow \subseteq P$. Since (r, P, c) is regular, by (d) and (a) we have that $cP_r^1 \subseteq P$. This yields Equation 4.6.1. By the same means we could show that $(s, Q, c) = (s, Q, c)(r, P, c)(s, Q, c)$. Thus (r, P, c) and (s, Q, c) are inverses of each other. \blacksquare

The Path Expansion for Semilattices

As we did for previous examples, we now turn to the path expansion of a semilattice and describe its properties. When the results and proofs are the same as in Proposition 4.6.1, we make note of this.

Proposition 4.6.2. *Let (S, Σ, f) be a semigroup system of a semilattice S . The following are true about $\mathcal{M}(S; \Sigma)$:*

- (a') (r, P, c) is an idempotent if and only if for any edge $(a, s) \in E(P_r^1)$, we have that $(c, s) \in E(P)$;
- (b') $E(\text{Path}(S; \Sigma))$ is a subsemigroup if and only if $|S| = 1$;
- (c') if elements (r, P, c) and (s, Q, d) commute, then $r = s$ and the following are true about P and Q
 - 1. if $c \notin V(Q)$, then $c < d$ and $d \in V(P)$ (no change);
 - 2. if $d \notin V(P)$, then $d < c$ and $c \in V(Q)$ (no change);
 - 3. if $d \in V(P)$, then $P \setminus P_d^\uparrow \subseteq Q$;

4. if $c \in V(Q)$, then $Q \setminus Q_c^\uparrow \subseteq P$;

(d') an element (r, P, c) is regular if and only if it is idempotent (no change);

(e') if (r, P, c) and (s, Q, d) are regular elements of $\text{Path}(S; \Sigma)$, then (r, P, c) has (s, Q, d) as an inverse if and only if $c = d$ and for any $t \in \Sigma$, we have $(c, t) \in E(P)$ if and only if $(c, t) \in E(Q)$.

Proof: We first prove part (a'). Let $(r, P, c) \in \text{Path}(S; \Sigma)$ be idempotent and let $(a, s) \in E(P_r^1)$. From the definition of elements in the path expansion, there is an $a \longrightarrow c$ path in P . Thus $c \leq a$, whereupon $ca = c$. From Proposition 4.6.1(a), we have that $cP_r^1 \subseteq P$. Combining these two facts yields, $(c, s) = c(a, s) \in E(cP_r^1) \subseteq E(P)$.

Conversely, suppose (r, P, c) is an element with the property that if $(a, s) \in E(P_r^1)$, then $(c, s) \in E(P)$. Let $(x, t) \in E(cP_r^1)$. This implies that $x \leq c$. Note that as we are in the path expansion, x can not be strictly less than c , since there must be a path from x to c . Thus $x = c$. From the original assumption, we know that $(x, t) = (c, t) \in E(P)$. We see that $cP_r^1 \subseteq P$, whereupon, from Proposition 4.6.1(a), the element (r, P, c) is idempotent.

The argument used for Proposition 4.6.1(b) applies equally well to (b'). We now show (c')3., noting that 4. can be shown in the same way. Let $(r, P, c) \in \text{Path}(S; \Sigma)$. If $x \in V(P)$ is such that x is accessible from c in $\text{Cay}(S; \Sigma)$, then $x = c$. Hence $P_c^\uparrow = P_c^{\uparrow\uparrow}$. The desired result then follows from Proposition 4.6.1(c)3..

Finally we look at (e'). In a semilattice, if digraphs P and Q are traversed by paths $rf \longrightarrow c$ and $sf \longrightarrow c$ respectively, then the property that $P_c^\uparrow = Q_c^\uparrow$ is equivalent to the property that $(c, t) \in E(P)$ if and only if $(c, t) \in E(Q)$ for any $t \in \Sigma$. Thus (e') follows from Proposition 4.6.1(e). ■

Chapter 5

Properties of $\mathcal{M}(S; \Sigma)$ and Properties of S

In this chapter we study how the properties of a semigroup system are reflected in its graph expansion. In Section 5.1 we will consider the connection between special elements, in particular periodic elements. This study of periodic elements can immediately be put to use in Section 5.2, where our goal is to show that if a semigroup S is E -dense, then S and $\mathcal{M}(S; \Sigma)$ share the same maximal group image.

In Sections 5.3 - 5.5 we discuss different properties of the semigroup graph expansion related to finiteness. First, we show in Section 5.3 that graph expansions preserve residual finiteness. Then in Section 5.4 we show that $\mathcal{M}(S, \Sigma, f)$ is finitely generated if and only if S and Σ are both finite. This contrasts with the path expansion, which we showed to be Σ -generated in Proposition 3.3.4(c). Next in Section 5.5, we show that the semigroup graph expansion has the same finite subgroup structure as the semigroup being expanded.

In the final section 5.6, we examine what happens to a subsemigroup T of semigroup S if we expand a system for S . Choosing any generating set, we prove that the graph expansion of a semigroup system for T maps to a graph expansion for any semigroup system of S .

5.1 Periodic, Idempotent, Regular, and Zero Elements

We are interested here in the relationship between the properties of elements of S and those of elements of $\mathcal{M}(S; \Sigma)$. One direction of this relationship is obvious: if (r, P, c) is periodic, idempotent, or a zero, then c is as well. Thus our focus here is on how knowing something about $c \in S$ provides information about $(r, P, c) \in \mathcal{M}(S; \Sigma)$. We first prove a general result about periodic elements, and then apply it to special cases like idempotent elements, regular elements, and zeros.

Proposition 5.1.1. *Suppose $(r, P, c) \in \mathcal{M}(S; \Sigma)$ and there exist some $m, n \in \mathbb{N}$ such that $c^m = c^{m+n}$. Then $(r, P, c)^{m+n} = (r, P, c)^{m+2n}$.*

Proof: Observe that c^m is the chosen vertex of both $(r, P, c)^m$ and $(r, P, c)^{m+n}$. Thus, right multiplying both these elements by $(r, P, c)^n$ will add the same edges and vertices to their respective subdigraphs. Obviously, these edges and vertices are contained in the subdigraph of $(r, P, c)^{m+n}$. Thus, $(r, P, c)^{m+n} = (r, P, c)^{m+2n}$. ■

Recall that an element $c \in S$ is aperiodic if there is a $k \in \mathbb{N}$ such that $c^k = c^{k+1}$. Using the value $n = 1$ in Proposition 5.1.1 yields the following corollary:

Corollary 5.1.2. *Suppose $(r, P, c) \in \mathcal{M}(S; \Sigma)$ and that c is aperiodic. Then (r, P, c) is aperiodic.*

The following result about idempotents is also a consequence of Proposition 5.1.1. However, we include an alternative proof to illustrate algebraically how the subdigraphs are being absorbed.

Corollary 5.1.3. *Suppose $(r, P, c) \in \mathcal{M}(S; \Sigma)$ and that c is an idempotent. Then $(r, P, c)^2$ is an idempotent of $\mathcal{M}(S; \Sigma)$.*

Proof: Let c be an idempotent of S .

$$\begin{aligned}
((r, P, c)^2)^2 &= (r, P \cup cP_r^1 \cup cP_r^1 \cup cP_r^1, c) \\
&= (r, P \cup cP_r^1, c) \\
&= (r, P, c)^2. \quad \blacksquare
\end{aligned}$$

Proposition 5.1.1 can also be used as a tool to construct regular elements in $\mathcal{M}(S; \Sigma)$ from regular elements in S .

Corollary 5.1.4. *Suppose that $(r, P, c), (s, Q, d) \in \mathcal{M}(S; \Sigma)$, that c is regular element of S , and that d is an inverse of c . Then $(r, P, c)(s, Q, d)(r, P, c)$ is regular.*

Proof: Assume the hypotheses. We use the symbol X to denote the product:

$$((r, P, c)(s, Q, d)(r, P, c)) (s, Q, d) ((r, P, c)(s, Q, d)(r, P, c)).$$

We want to show that X equals $(r, P, c)(s, Q, d)(r, P, c)$. Our assumptions imply that dc is idempotent. Thus, we can use Proposition 5.1.1 to simplify X .

$$\begin{aligned}
X &= (r, P, c)((s, Q, d)(r, P, c))^3 \\
&= (r, P, c)((s, Q, d)(r, P, c))^2 \\
&= (r, P \cup cQ_s^1 \cup cdP_r^1 \cup cdcQ_s^1 \cup cdcdP_r^1, cdcdc) \\
&= (r, P \cup cQ_s^1 \cup cdP_r^1, c) \\
&= (r, P, c)(s, Q, d)(r, P, c). \quad \blacksquare
\end{aligned}$$

In the next Proposition, we show how the presence of a zero in S influences $\mathcal{M}(S; \Sigma)$.

Proposition 5.1.5. *Let (S, Σ, f) be a semigroup system of a semigroup S with zero, denoted 0 . The following are true about $\mathcal{M}(S; \Sigma)$:*

- (a) *Assume $|\Sigma| < \infty$. An element (r, P, c) is a left-zero of $\mathcal{M}(S; \Sigma)$ if and only if $c = 0$ and for every $t \in \Sigma$, $(0, t) \in E(P)$;*
- (b) *$\mathcal{M}(S; \Sigma)$ has no left-zeros if Σ is infinite;*
- (c) *the following are equivalent:*
 - i. *$\mathcal{M}(S; \Sigma)$ has a right-zero;*
 - ii. *$\mathcal{M}(S; \Sigma)$ has a zero;*
 - iii. *$|\Sigma| = 1$.*

Proof: For part (a), suppose (r, P, c) is a left-zero. Let $t \in \Sigma$. Then

$$(r, P, c) = (r, P, c)(t, \bullet_t, tf) = (r, P \cup \{(c, t)\}, c). \quad (5.1.1)$$

There is some word $t_1 \dots t_n \in \Sigma^+$ (with each $t_i \in \Sigma$) such that $(t_1 \dots t_n)f = 0$. Since $c(tf) = c$ for all $t \in \Sigma$, $c = c(t_1 \dots t_n)f = c(0) = 0$. We also see from Equation 5.1.1 that $(0, t) \in E(P)$ for every $t \in \Sigma$. For the converse, suppose that the digraph P in $(r, P, 0)$ has the property that for all $t \in \Sigma$, $(0, t) \in E(P)$. Then $(r, P, 0)(t, \bullet_t, tf) = (r, P \cup \{(0, t)\}, 0) = (r, P, 0)$. We see immediately that $(r, P, 0)$ is a left-zero of $\mathcal{M}(S; \Sigma)$.

Moving on to (b), we suppose Σ is infinite. Then there is no finite subdigraph of P that contains every edge of the form $(0, t)$ for all $t \in \Sigma$. Hence $\mathcal{M}(S; \Sigma)$ does not contain any left-zeros.

For (c), the implication ii. \Rightarrow i. is clear. We now show i. \Rightarrow iii. Suppose (r, P, c) is a right-zero and let $(s, Q, d) \in \mathcal{M}(S; \Sigma)$. Then

$$(r, P, c) = (s, Q, d)(r, P, c) = (s, Q \cup dP_r^1, dc).$$

From this we see that $r = s$. Thus, $|\Sigma| = 1$.

We conclude by showing iii. \Rightarrow ii. Assume $\Sigma = \{r\}$. Since S contains a zero, there is some $n \in \mathbb{N}$ such that $(rf)^n = 0$. This means that S is finite and hence its Cayley graph is as well. Let $P = \text{Cay}(S; \{r\})$. We claim that the element $(r, P, 0)$ is the zero of $\mathcal{M}(S; \{r\})$. From (a), we know that it is a left-zero. Let $(r, Q, d) \in \mathcal{M}(S; \{r\})$. Since there is exactly one generator, $Q \cup dP_r^1 = P$. Thus, $(r, Q, d)(r, P, 0) = (r, Q \cup dP_r^1, 0) = (r, P, 0)$. Since $(r, P, 0)$ is a right-zero, we conclude that it is a zero. \blacksquare

5.2 A Shared Maximal Group Image

One of the special properties of the group graph expansion is that it is an E -unitary inverse semigroup. In contrast, we will start this section by proving that the semigroup graph expansion is never E -unitary. This should not come as a surprise, since none of the examples given in Chapter 4 was E -unitary. This established, our second goal will be to consider how the E -unitary property can be replaced in a way that applies to semigroup graph expansion. This will lead us to show that for E -dense semigroups, the semigroup graph expansion has the same maximal group image as the original semigroup.

Proposition 5.2.1. *Let (S, Σ, f) be a semigroup system. Then $\mathcal{M}(S; \Sigma)$ is not E -unitary.*

Proof: In our definition of E -unitary, we exclude all semigroups that do not have idempotents. Thus, when showing that $\mathcal{M}(S; \Sigma)$ is not E -unitary, we only need to consider the case when $E(\mathcal{M}(S; \Sigma)) \neq \emptyset$. Let $(r, P, c) \in E(\mathcal{M}(S; \Sigma))$. From Theorem 3.2.1 (b), $cP_r^1 \subseteq P$, whereupon $(c, r) \in E(P)$. We wish to construct an element which is a right identity of (r, P, c) , but which is not idempotent. Let Q be the digraph $Q = (P \setminus \{(c, r)\})_{rf}^\dagger$. Note that $c \in V(Q)$, whereupon $(r, Q, c) \in \mathcal{M}(S; \Sigma)$. However, we have that $cQ_r^1 \not\subseteq Q$, from which we know that (r, Q, c) is not idempotent. But,

$cQ_r^1 \subseteq cP_r^1 \subseteq P$, from which we have:

$$(r, P, c)(r, Q, c) = (r, P \cup cQ_r^1, c) = (r, P, c).$$

Thus $\mathcal{M}(S; \Sigma)$ is not E -unitary. ■

A semigroup S is E -dense if for every $x \in S$, there exists a $y \in S$ such that xy is an idempotent. Many familiar semigroups are E -dense including all groups, inverse semigroups, periodic semigroups, and finite semigroups. Moreover, E -dense semigroups have maximal group images just like E -unitary inverse semigroups do. This result was shown by Hall and Munn in [9]. In this thesis, we included it as Lemma 2.2.1. We can use our previous results about the idempotents of $\mathcal{M}(S; \Sigma)$ to show that the graph expansion construction preserves and reflects E -density.

Proposition 5.2.2. *Let (S, Σ, f) be a semigroup system of a semigroup S . Then $\mathcal{M}(S; \Sigma)$ is E -dense if and only if S is E -dense.*

Proof: Suppose S is E -dense. Let $(r, P, c) \in \mathcal{M}(S; \Sigma)$. Since S is E -dense, there exists some $d \in S$ such that cd is idempotent. Let $(s, Q, d) \in \mathcal{M}(S; \Sigma)$. From Corollary 5.1.3, $(r, P, c)((s, Q, d)(r, P, c)(s, Q, d)) = ((r, P, c)(s, Q, d))^2$ is idempotent. Conversely, suppose $\mathcal{M}(S; \Sigma)$ is E -dense. Since S is a homomorphic image of $\mathcal{M}(S; \Sigma)$, S is E -dense. ■

Applying Hall and Munn's result in Lemma 2.2.1 to Proposition 5.2.2 shows that if S is E -dense, then the graph expansion $\mathcal{M}(S; \Sigma)$ is E -dense and thus has a maximal group image as well. We wish to show that this is the same maximal group image as that of S . In order to do so, we will use the following Lemma:

Lemma 5.2.3. *Let (S, Σ, f) be a semigroup system of an E -dense semigroup S . Let $(r, P, c), (s, Q, c) \in \mathcal{M}(S; \Sigma)$. If H is a group and there exists a homomorphism $\alpha : \mathcal{M}(S; \Sigma) \rightarrow H$, then $(r, P, c)\alpha = (s, Q, c)\alpha$.*

Proof: Let $(r, P, c)\alpha = h$ and $(s, Q, c)\alpha = h'$. Because S is E -dense, there exists some $d \in S$ such that cd is an idempotent. Let (x, T, d) be any element with d as the chosen vertex. Define (x, A, dcd) to be the product:

$$(x, T, d)(r, P, c)(x, T, d)(s, Q, c)(x, T, d).$$

Let $(x, A, dcd)\alpha = k$. We claim that the product $(r, P, c)(x, A, dcd)$ is idempotent. To see this, note that

$$(r, P, c)(x, A, dcd) = (r, P \cup cT_x^1 \cup cdP_r^1 \cup cdcT_x^1 \cup cdQ_s^1, cd).$$

Since cd is idempotent and

$$cd(P \cup cT_x^1 \cup cdP_r^1 \cup cdcT_x^1 \cup cdQ_s^1) = cdcT_x^1 \cup cdP_r^1 \cup cdQ_s^1,$$

by Theorem 3.2.1 (b), we know that the product $(r, P, c)(x, A, dcd)$ is idempotent. Using a similar argument, $(s, Q, c)(x, A, dcd)$ is idempotent too.

Every group homomorphism sends idempotents to the identity. Thus,

$$((r, P, c)(x, A, dcd))\alpha = ((s, Q, c)(x, A, dcd))\alpha = 1,$$

whereupon we have that $hk = 1$ and $h'k = 1$. This implies that $h = h'$, as desired. ■

Lemma 5.2.3 implies that if S is an E -dense semigroup, then any homomorphism from $\mathcal{M}(S; \Sigma)$ to a group H factors uniquely through S . This gives us the desired result:

Theorem 5.2.4. *Let (S, Σ, f) be a semigroup system of an E -dense semigroup S . Then S and $\mathcal{M}(S; \Sigma)$ have the same maximal group image.*

Proof: Since S is E -dense, it follows from Proposition 5.2.2 that $\mathcal{M}(S; \Sigma)$ is E -dense. Thus both S and $\mathcal{M}(S; \Sigma)$ possess maximal groups, G and G' respectively.

In the diagram, H represents any group for which there is a homomorphism $\alpha : \mathcal{M}(S; \Sigma) \rightarrow H$. Lemma 5.2.3 guarantees the existence of β satisfying $\alpha = \epsilon_S \circ \beta$. Since ϵ_S is surjective, β is necessarily unique.

5.3 Residual Finiteness

We wish to show that the semigroup graph expansion preserves residual finiteness. Recall, a semigroup S is residually finite if for every $x, y \in S$, there exists some semigroup T and a map $\alpha : S \rightarrow T$ such that $x\alpha \neq y\alpha$. To aid in the proof, we define the *set of factors* of an element $x \in S$ to be the following subset of S :

$$\text{Fact}(x) = \{ a \mid \text{there exist some } b, c \in S^1 \text{ such that } bac = x \}.$$

We will use the following result:

Lemma 5.3.1. *Let S be a semigroup. If $\text{Fact}(x)$ is finite for every $x \in S$, then S is residually finite.*

Proof: Let x and y be distinct elements of S . The set $I = S \setminus \{\text{Fact}(x) \cup \text{Fact}(y)\}$ is an ideal of S . Thus we form the Rees quotient, $\mathcal{M}(S; \Sigma)/I$, which is equal as a set to $\mathcal{M}(S; \Sigma) \setminus I \cup \{0\}$. If b and c are elements of $\mathcal{M}(S; \Sigma)/I$, their product is defined:

$$bc = \begin{cases} bc & \text{if } bc \notin I \\ 0 & \text{otherwise.} \end{cases}$$

Note that the images of x and y are distinct. Thus S is residually finite. ■

Lemma 5.3.2. *Let (S, Σ, f) be a semigroup system. If S is residually finite, then $\mathcal{M}(S; \Sigma)$ is residually finite.*

Proof: Suppose S is residually finite. Let $(r, P, c) \in \mathcal{M}(S; \Sigma)$. We wish to show that $\text{Fact}(r, P, c)$ is finite. First, let $\Gamma_P \subseteq \text{Cay}(S; \Sigma)$ be the maximal digraph with $V(\Gamma_P) = \bigcup_{v \in V(P)} \text{Fact}(v)$ and $\Sigma(\Gamma_P) \subseteq \Sigma(P) \cup \{r\}$. The finiteness of each of the factor

sets $\text{Fact}(v)$ and the edge color set $\Sigma(P)$ imply that Γ_P is finite. By construction, if $A \subseteq \text{Cay}(S; \Sigma)$ and $v \in P$ is such that subdigraph such that $vA \subseteq P$, then $A \subseteq \Gamma_P$.

Suppose $(s, Q, d) \in \text{Fact}(r, P, c)$. It follows that $s \in \Sigma_P \cup \{r\}$, $Q \subseteq \Gamma_P$, and $d \in \text{Fact}(c)$. Since $\Sigma_P \cup \{r\}$ and $\text{Fact}(c)$ are finite sets and Γ_P is a finite digraph, there are only finitely many such (s, Q, d) . Thus $\text{Fact}(r, P, c)$ is finite, whereupon Lemma 5.3.1 gives that $\mathcal{M}(S; \Sigma)$ is residually finite. ■

5.4 Generating Sets

Clearly if S is finite, then $\mathcal{M}(S; \Sigma)$ is finite, and vice versa. In this section, we will give criteria on S for $\mathcal{M}(S; \Sigma)$ to be finitely generated. We will be using the description of indecomposable elements given in Lemma 3.2.2.

Theorem 5.4.1. *$\mathcal{M}(S; \Sigma)$ is finitely generated if and only if S and Σ are finite.*

Proof: Clearly if S and Σ are finite, then $\mathcal{M}(S; \Sigma)$ is finite and hence finitely generated. Conversely, suppose $\mathcal{M}(S; \Sigma)$ is finitely generated. By Lemma 3.2.2(b), all elements in the set $\{(s, \bullet_{sf}, sf) \mid s \in \Sigma\}$ are indecomposable and hence in any generating set for $\mathcal{M}(S; \Sigma)$. Thus, Σ must be finite. Assume, by way of contradiction, that S is infinite. Then for any number $n \in \mathbb{N}$, there exists an element of S whose minimal length when expressed as a word in Σ^+ is greater than or equal to n . Thus we can find a set of minimal length representatives $\{w_1, w_2, \dots\}$ for elements of S such that

$$2 \leq |w_1| < |w_2| < |w_3| < \dots$$

We express each w_i as $r_i v_i$ where $r_i \in \Sigma$, $v_i \in \Sigma^+$. Because each w_i is a minimal length representative, there are no words $s_i \in \Sigma^+$ and $t_i \in \Sigma^*$, with $w_i = r_i s_i t_i$, such that $r_i = r_i s_i$. Consequently $[r_i f \xrightarrow{v_i} w_i f]$ contains no cycles passing through $r_i f$. Hence, appealing to Lemma 3.2.2(b), the element $(r_i, [r_i f \xrightarrow{v_i} w_i f], r_i f)$ is indecomposable. Therefore the set $\{(r_i, [r_i f \xrightarrow{v_i} w_i f], r_i f) \mid i \in \mathbb{N}\}$ contains an infinite number of indecomposable elements, contradicting the fact that $\mathcal{M}(S; \Sigma)$ is finitely generated. We

conclude that S is finite. ■

This results distinguish the semigroup graph expansion from the group graph expansion $M_{gp}(G; \Omega)$ which is generated as an inverse semigroup by $\{(\bullet_1 \longrightarrow \bullet_g, gf) | g \in \Omega\}$. With respect to generation, $\text{Path}(S; \Sigma)$, which we showed in Proposition 3.3.4(c) to be generated by $\{(s, \bullet_s, sf) | s \in \Sigma\}$, is a closer analogue of the group graph expansion.

5.5 Subgroups of $\mathcal{M}(S; \Sigma)$ and S

We investigate the close relationship between the subgroups of $\mathcal{M}(S; \Sigma)$ and the finite subgroups of S .

Theorem 5.5.1. *Let (S, Σ, f) be a semigroup system of a semigroup S and let W be a subgroup of $\mathcal{M}(S; \Sigma)$. Then the following are true:*

- (a) *if $(r, P, c), (s, Q, d) \in W$, then $r = s$ and $P = Q$;*
- (b) *W is finite;*
- (c) *the homomorphism ϵ_S is injective when restricted to W ;*
- (d) *if T is a finite subgroup of S , there exists a subgroup $W' \leq \mathcal{M}(S; \Sigma)$ such that $W'\epsilon_S = T$.*

Proof: We start by proving part (a). Let W be a subgroup of $\mathcal{M}(S; \Sigma)$ containing elements (r, P, c) and (s, Q, d) . Suppose W has identity (a, A, e) . Then

$$(r, P, c) = (a, A, e)(r, P, c) = (a, A \cup eP_r^1, ec) \quad (5.5.1)$$

It is clear from Equation 5.5.1 that $r = a$. Similarly we could show that $s = a$, whereupon $r = s$. Let (r, T, b) be the inverse of (r, P, c) in W . Then

$$(a, A, e) = (r, P, c)(r, T, b) = (r, P \cup cT_r^1, cb). \quad (5.5.2)$$

From Equation 5.5.1 we see that $A \subseteq P$; from Equation 5.5.2 we see that $P \subseteq A$. Combining this information produces $A = P$. Similarly, we could show $Q = A$, whereupon $P = Q$.

Applying the result (a) to (b), we see that $|W| \leq |V(P)|$. The digraph P is by definition finite. Thus W is a finite subgroup.

Now we consider part (c). It is clear that the homomorphism ϵ_S , when restricted to W , is injective, because the elements of W are distinguished by their chosen vertices, and thus will be sent by ϵ_S to distinct elements of S .

For part (d), our approach will be to construct W' . Let $e \in T$ denote the identity element in T . Choose a finite subset $\Upsilon \subseteq \Sigma^+$ which contains a word w for which $wf = e$ and which generates T (i.e., Υf generates T as a semigroup). Write w as $w = rv$, where $r \in \Sigma$ and $v \in \Sigma^*$. We construct a subdigraph P by taking the union of the following:

1. Include the digraph $[rf \xrightarrow{v} e]$;
2. For each $c \in T$ and $u \in \Upsilon$, include $[c \xrightarrow{u} c(uf)]$. Note that $c(uf) \in T$, since $uf \in T$.

Due to its construction, P has the following properties:

- P is a finite subdigraph;
- P is rooted at rf ;
- $T \subseteq V(P)$;
- for any $c \in T$, $cP_r^1 \subseteq P$.

To see the last property, we will show that $E(cP_r^1) \subseteq E(P)$. First consider the edge (c, r) . This is clearly in $E(P)$, because $E(P)$ contains the path $c \xrightarrow{rv} c((rv)f)$. Next, consider any edge $(x, s) \in E(cP)$. We can rewrite x as cx' . There is a vertex $d \in V(P)$ and a word $u \in \Upsilon$ such that the edge (x', s) lies on the path $d \xrightarrow{u} d(uf)$. Since $c, d \in T$ and T is a subgroup, $cd \in T$, which implies that P contains the path

$cd \xrightarrow{u} cd(uf)$. The edge $c \cdot (x', s) = (cx', s) = (x, s)$ lies on this path and is thus in $E(P)$.

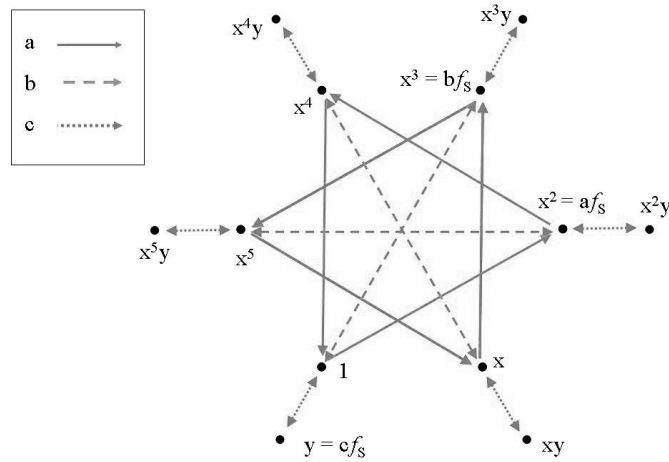
For P and r as specified, define $W' = \{(r, P, c) | c \in T\}$. We would like to show that W' is a subgroup of $\mathcal{M}(S; \Sigma)$. First, let $(r, P, c), (r, P, d) \in W'$. Then

$$(r, P, c)(r, P, d) = (r, P \cup cP_r^1, cd) = (r, P, cd).$$

As mentioned before, since T is a subgroup, $cd \in T$. Thus we have $(r, P, cd) \in W'$. The element (r, P, e) is the identity of W' ; an element (r, P, c) has inverse (r, P, c^{-1}) , where c and c^{-1} are inverses in T . We conclude that W' is a subgroup of $\mathcal{M}(S; \Sigma)$. Clearly it projects onto T . ■

We provide an example of this construction.

Example: Consider the dihedral group $D_6 = gp\langle x, y | x^6 = y^2 = (xy)^2 = 1 \rangle$. Let $\Sigma = \{a, b, c\}$ and define the map $f : \Sigma \rightarrow S$ by $af = x^2$, $bf = x^3$, and $cf = y$. From this we form the graph expansion $\mathcal{M}(D_6, \Sigma)$. Let $T < S$ be the subgroup generated by $\{x\}$. Observe that T is isomorphic to the cyclic group $C_6 = \{z | z^6 = 1\}$. Choose $\Upsilon = \{c^2, a^2b\}$. Note that Υf generates T since $(a^2b)f = x^7 = x$. Similarly Υ contains a word corresponding to the identity of T , namely $(c^2)f = y^2 = 1$. Using Υ , we form the subdigraph P of $\text{Cay}(S; \Sigma)$, shown below:



For this example, $W' = \{(c, P, x^i) | 0 \leq i \leq 5\}$. ■

5.6 Subsemigroups of S and $\mathcal{M}(S; \Sigma)$

In this section, we consider how subsemigroups are preserved by the semigroup graph expansion. We start with a subsemigroup T of a semigroup S . Let (T, Ψ, g) and (S, Σ, f) be respective semigroup systems. Our goal will be to describe the relationship between $\mathcal{M}(T; \Psi)$ and $\mathcal{M}(S; \Sigma)$. For the moment, we will not assume any relationship between Ψ and Σ . However, since T embeds in S , we can find a function $\beta : \Psi^+ \rightarrow \Sigma^+$ for which $g = \beta \circ f$. (In fact, if f sends multiple words in Σ^+ to the same element in T , there may be many possibilities for β .) The following notation will help us work with elements: if $w \in \Psi$, we will denote $w\beta$ by \hat{w} . Additionally, we will sometimes write each \hat{w} as $\hat{r}\hat{v}$, where $\hat{r} \in \Sigma$ and $\hat{v} \in \Sigma^*$. The maps $T \hookrightarrow S$ and $\beta : \Psi^+ \rightarrow \Sigma^+$ form a semigroup system homomorphism between (T, Ψ, g) and (S, Σ, f) .

We want to define a homomorphism from $\mathcal{M}(T; \Psi)$ to $\mathcal{M}(S; \Sigma)$. In order to do so, we must first determine how to map rooted subdigraphs of $\text{Cay}(T; \Psi)$ to rooted subdigraphs of $\text{Cay}(S; \Sigma)$. Recall that $\mathcal{P}(\text{Cay}(T; \Psi))$ is the set of subdigraphs of $\text{Cay}(T; \Psi)$ and it is a semigroup with the operation of union. We can thus define a function $\hat{\beta} : \mathcal{P}(\text{Cay}(T; \Psi)) \rightarrow \mathcal{P}(\text{Cay}(S; \Sigma))$ by $P \mapsto P'$ where $V(P') = V(P)$ and

$$E(P') = \left\{ (x, s) \left| \begin{array}{l} \text{where there exists some } (c, w) \in E(P) \\ \text{and } (x, s) \text{ lies on the path } c \xrightarrow{\hat{w}} c(wf) \end{array} \right. \right\}$$

The idea behind the map $\hat{\beta} : P \rightarrow P'$ is that the edge $(x, w) \in E(P)$ is mapped to the underlying graph of the path $[x \xrightarrow{\hat{w}} x(wf)] \in P'$. We give properties of $\hat{\beta}$ in the next lemma.

Lemma 5.6.1. *Let T be a subsemigroup of a semigroup S and let (T, Ψ, g) and (S, Σ, f) be semigroup systems, and let $\hat{\beta} : \mathcal{P}(\text{Cay}(T; \Psi)) \rightarrow \mathcal{P}(\text{Cay}(S; \Sigma))$ be as described above. Then:*

1. $\hat{\beta}$ is a semigroup homomorphism;
2. $\hat{\beta}$ preserves the left action of elements of T on digraphs, i.e. for $c \in T$ and $P \subseteq \text{Cay}(T; \Psi)$, we have $(cP)\hat{\beta} = c(P\hat{\beta})$.

Proof: For (a), we show that if $P, Q \subseteq \text{Cay}(T; \Psi)$, then $(P \cup Q)\hat{\beta} = P\hat{\beta} \cup Q\hat{\beta}$. From the definition of $\hat{\beta}$,

$$V((P \cup Q)\hat{\beta}) = V(P \cup Q) = V(P) \cup V(Q) = V(P\hat{\beta}) \cup V(Q\hat{\beta}).$$

Let $(x, s) \in E((P \cup Q)\hat{\beta})$. Then there exists some $(y, w) \in E(P \cup Q)$ such that the edge (x, s) lies on the path $y \xrightarrow{w\beta} y(wf)$. If $(y, w) \in E(P)$, then $(x, s) \in E(P\hat{\beta})$. Similarly, if $(y, w) \in E(Q)$, then $(x, s) \in E(Q\hat{\beta})$. Thus we have $(x, s) \in E(P\hat{\beta}) \cup E(Q\hat{\beta})$, which implies that $E((P \cup Q)\hat{\beta}) \subseteq E(P\hat{\beta}) \cup E(Q\hat{\beta})$. The reverse inclusion can be shown by reversing the steps. We conclude that $E((P \cup Q)\hat{\beta}) = E(P\hat{\beta}) \cup E(Q\hat{\beta})$, whereupon $(P \cup Q)\hat{\beta} = P\hat{\beta} \cup Q\hat{\beta}$.

Proceeding to (b), let $c \in T$ and $P \subseteq \text{Cay}(T; \Psi)$. Clearly,

$$V((cP)\hat{\beta}) = V(cP) = cV(P) = cV(P\hat{\beta}) = V(c(P\hat{\beta})).$$

Next, let $(x, s) \in E((cP)\hat{\beta})$. Then there exists some $(y, w) \in E(cP)$ such that the edge (x, s) lies on the path $y \xrightarrow{w\beta} y(wf)$. Further there exists a $z \in V(P)$ such that $y = cz$ and $(z, w) \in E(P)$. The map $\hat{\beta}$ sends (z, w) to $\lfloor z \xrightarrow{w} z(wf) \rfloor$. Thus,

$$\begin{aligned} \lfloor y \xrightarrow{w\beta} y(wf) \rfloor &= \lfloor cz \xrightarrow{w} cz(wf) \rfloor \\ &= c(\lfloor z \xrightarrow{w} z(wf) \rfloor) \\ &= c(z, w)\hat{\beta} \\ &\subseteq c(P\hat{\beta}). \end{aligned}$$

Since the digraph $\lfloor y \xrightarrow{w\beta} y(wf) \rfloor$ is contained in $c(P\hat{\beta})$, it follows that the edge $(x, s) \in E(c(P\hat{\beta}))$. Thus $E((cP)\hat{\beta}) \subseteq E(c(P\hat{\beta}))$. The reverse inclusion can be shown by reversing the steps. We conclude that $E((cP)\hat{\beta}) = E(c(P\hat{\beta}))$, whereupon $(cP)\hat{\beta} = c(P\hat{\beta})$. ■

Again, it will be convenient to factor the image \widehat{w} of an element $w \in \Psi$ as $\widehat{w} = \widehat{r_w} \widehat{v_w}$ where $\widehat{r_w} \in \Sigma$ and $\widehat{v_w} \in \Sigma^*$. We now use the maps $\beta : \Psi^+ \rightarrow \Sigma^+$ and $\widehat{\beta} : \mathcal{P}(\text{Cay}(T; \Psi)) \rightarrow \mathcal{P}(\text{Cay}(S; \Sigma))$ to define a map, denoted by κ , between the graph expansions:

$$\begin{aligned} \kappa : \mathcal{M}(T; \Psi) &\rightarrow \mathcal{M}(S; \Sigma) \\ (w, P, c) &\mapsto (\widehat{r_w}, [\widehat{r_w} f \xrightarrow{\widehat{v_w}} wg] \cup P \widehat{\beta}, c). \end{aligned}$$

Theorem 5.6.2. *Let (S, Σ, f) and (T, Ψ, g) be semigroup systems of a semigroup S which has subsemigroup T . Further let the maps β , $\widehat{\beta}$, and κ be as described above. Then κ is a homomorphism for which the diagram below commutes.*

$$\begin{array}{ccc} \mathcal{M}(T; \Psi) & \xrightarrow{\kappa} & \mathcal{M}(S; \Sigma) \\ \varepsilon_T \downarrow & & \downarrow \varepsilon_S \\ T & \xrightarrow{id} & S \end{array}$$

Proof: We first show that κ is a homomorphism. To this end, let

$$(x, P, c), (y, Q, d) \in \mathcal{M}(T; \Psi).$$

$$\begin{aligned} (x, P, c) \kappa(y, Q, d) \kappa &= (\widehat{r_x}, [\widehat{r_x} f \xrightarrow{\widehat{v_x}} xg] \cup P \widehat{\beta}, c) (\widehat{r_y}, [\widehat{r_y} f \xrightarrow{\widehat{v_y}} yg] \cup Q \widehat{\beta}, d) \\ &= (\widehat{r_x}, [\widehat{r_x} f \xrightarrow{\widehat{v_x}} xg] \cup P \widehat{\beta} \cup (c, \widehat{r_y}) \cup c[\widehat{r_y} f \xrightarrow{\widehat{v_y}} yg] \cup c(Q \widehat{\beta}), cd) \\ &= (\widehat{r_x}, [\widehat{r_x} f \xrightarrow{\widehat{v_x}} xg] \cup P \widehat{\beta} \cup [c \xrightarrow{y\beta} yg] \cup c(Q \widehat{\beta}), cd) \\ &= (\widehat{r_x}, [\widehat{r_x} f \xrightarrow{\widehat{v_x}} xg] \cup P \widehat{\beta} \cup c((1, y) \widehat{\beta}) \cup c(Q \widehat{\beta}), cd). \end{aligned}$$

From Lemma 5.6.1, we can replace $P \widehat{\beta} \cup c((1, y) \widehat{\beta}) \cup c(Q \widehat{\beta})$ by $(P \cup cQ_y^1) \widehat{\beta}$. Thus we continue:

$$\begin{aligned} (x, P, c) \kappa(y, Q, d) \kappa &= (\widehat{r_x}, [\widehat{r_x} f \xrightarrow{\widehat{v_x}} xg] \cup (P \cup cQ_y^1) \widehat{\beta}, cd) \\ &= (x, P \cup cQ_y^1, cd) \kappa \\ &= (x, P, c)(y, Q, d) \kappa. \end{aligned}$$

We conclude κ is a homomorphism. Since κ is the identity map on the third coordinate, the diagram clearly commutes. \blacksquare

Theorem 5.6.2 sheds light on the relationship between the semigroup graph expansions of two different semigroup systems of the same semigroup.

Corollary 5.6.3. *Let (S, Ψ, g) and (S, Σ, f) be semigroup systems of a semigroup S and let $id : S \rightarrow S$, $\beta : \Psi^+ \rightarrow \Sigma^+$ be a semigroup system homomorphism between them. Further, let $\kappa : \mathcal{M}(S, \Psi) \rightarrow \mathcal{M}(S, \Sigma)$ be as described above. Then κ is surjective if and only if β is surjective.*

Proof: Suppose κ is surjective. Let $r \in \Sigma$. There exists an element $(w, P, c) \in \mathcal{M}(T; \Psi)$ such that

$$(w, P, c)\kappa = (r_w, [\widehat{r_w}f \xrightarrow{v_w} wg] \cup P\widehat{\beta}, c) = (r, \bullet_r, rf).$$

We see that $\widehat{r_w} = r$ and $v_w = \epsilon$. Hence $r = \widehat{r_w} = w\beta$. We conclude that $\Sigma \subseteq \Psi\beta$, whereupon we have that β is surjective.

Conversely, suppose that β is surjective. Let $(r, P, c) \in \mathcal{M}(S, \Sigma)$. We decompose the digraph P into the union of a finite number of rf -rooted paths $rf \xrightarrow{w_i} (rw_i)f$ with $i \in \Phi$ where Φ is an index set. Since β is surjective, for each $r \in \Sigma$ we can find a preimage $r' \in \Psi$ such that $r'\beta = r$. This implies that $r'g = r'\beta \circ f$. Let w'_i denote the word formed by the concatenation of the preimages of the letters in w . From the definition of κ , $[r'g \xrightarrow{w'_i} (r'w'_i)g]\kappa = [rf \xrightarrow{w_i} (rw_i)f]$. Thus we have that

$$\left(\bigcup_{i \in \Phi} [\bullet_{r'g} \xrightarrow{w'_i} \bullet_{(r'w'_i)g}] \right) \kappa = \bigcup_{i \in \Phi} [\bullet_{rf} \xrightarrow{w_i} \bullet_{(rw_i)f}] = (r, P, c). \quad \blacksquare$$

Chapter 6

Green's Relations

In this chapter, we investigate Green's relations for the graph expansions of semigroups. Our approach is to look at each relation and describe the structure of elements that belong to the same class. These results generalize Margolis and Meakin's findings about the \mathcal{R} -, \mathcal{L} -, \mathcal{H} -, \mathcal{J} -, and \mathcal{D} -classes for group graph expansions. In contrast to the group setting where these results can be deduced from straightforward observations, the semigroup versions often require involved proofs. For comparison, we state Margolis and Meakin's results here:

Lemma 6.0.4 (Lemma 3.2 of [17]). *For $(P, c), (Q, d) \in M_{gp}(G; \Omega)$ we have the following:*

- (a) $(P, c)\mathcal{R}(Q, d)$ if and only if $P = Q$;
- (b) $(P, c)\mathcal{L}(Q, d)$ if and only if $c^{-1}P = d^{-1}Q$;
- (c) The maximal subgroup $H_{(P,1)}$ of $M_{gp}(G; \Omega)$ is equal to the label-preserving automorphism group of the graph P and is also isomorphic to $\text{stab}(P) = \{h \in G \mid h \cdot P = P\}$;
- (d) $(P, c)\mathcal{J}(Q, d)$ if and only if P and Q are isomorphic as labeled graphs;
- (e) Every \mathcal{J} -class of $M_{gp}(G; \Omega)$ is finite;
- (f) $\mathcal{D} = \mathcal{J}$.

The structure of this chapter is as follows: we will start by characterizing the \mathcal{R} -classes. These are very simple to describe using a few straightforward observations. We then proceed to the \mathcal{L} -classes. Their description, compared to the \mathcal{R} -classes, is much more involved. Thus, upon providing a general description, we look at how the \mathcal{L} -class description can be simplified for specific types of semigroups. Next, we use the techniques developed for the \mathcal{L} - and \mathcal{R} -classes to characterize the \mathcal{H} - and \mathcal{D} -classes. We discuss how the structure of the \mathcal{H} -class of an element (r, P, c) is related to the automorphism group of the subdigraph P_c^\uparrow . In the last section, we prove that $\mathcal{D} = \mathcal{J}$ for semigroup graph expansions. At the end of the section, we consider when the graph expansion is finite- \mathcal{J} -above, to be defined below in Section 6.5.

Throughout this section we assume that (S, Σ, f) is a semigroup system and that $(r, P, c), (s, Q, d) \in \mathcal{M}(S; \Sigma)$. We also will be working frequently with maps between Cayley graphs. Trying not to add additional layers of notation, if $P, Q \subseteq \text{Cay}(S, X)$, $\theta : P \rightarrow Q$, and $v \in V(P)$, we use the notation $v\theta$ to indicate the label of the image of the vertex \bullet_v under θ . We now prove a very useful result about label-preserving mappings of subdigraphs of Cayley digraphs.

Lemma 6.0.5. *Let (S, Σ, f) be a semigroup system. Let $P \subseteq \text{Cay}(S; \Sigma)$ be an x -rooted subdigraph of $\text{Cay}(S; \Sigma)$ and let $a, y \in S$ be such that $y = ax$. If $\theta : P \rightarrow \text{Cay}(S; \Sigma)$ is a label-preserving graph map which sends x to y , then the map θ is the same as the map corresponding to translation by a .*

Proof: Let $v \in V(P)$. Since P is x -rooted, there exists a word $w \in \Sigma^*$ labeling a $x \rightarrow w$ path in P . Note that $v = x(wf)$. As θ is a label-preserving map, the path $x \xrightarrow{w} v$ is mapped by θ to the path $y \xrightarrow{w} y(wf)$. Thus $v\theta = y(wf) = ax(wf) = av$. Let $(v, r) \in E(P)$. Having assumed that θ is label-preserving and since translation is also label-preserving, we know that $(v, r)\theta = (v\theta, r) = (av, r) = a(v, r)$. We conclude that the map θ is the same as the map given by translation by a . ■

6.1 \mathcal{R} -Classes

We describe the \mathcal{R} -classes:

Theorem 6.1.1. *Assume $(r, P, c) \neq (s, Q, d)$. Then $(r, P, c)\mathcal{R}(s, Q, d)$ if and only if $r = s$, $P = Q$, and there is a cycle in P containing c and d . It follows that if $(r, P, c)\mathcal{R}(s, Q, d)$, then $c \mathcal{R} d$ in S .*

Proof: Suppose $(r, P, c)\mathcal{R}(s, Q, d)$ with $(r, P, c) \neq (s, Q, d)$. Then there exist some (x, A, a) , (y, B, b) such that

$$(r, P, c)(x, A, a) = (r, P \cup cA_x^1, ca) = (s, Q, d); \quad (6.1.1)$$

$$(s, Q, d)(y, B, b) = (s, Q \cup dB_y^1, db) = (r, P, c). \quad (6.1.2)$$

From Equation 6.1.1 we see that $r = s$, $P \subseteq Q$, and d is accessible from c . Similarly from Equation 6.1.2, we have that $Q \subseteq P$ and c is accessible from d . The accessibility results imply that there is cycle in P containing both c and d .

Conversely, assume that $r = s$, $P = Q$, and there is a cycle in P containing both c and d . We can find a word $w \in \Sigma^+$ that labels a $c \rightarrow d$ path in P . Write the word as $w = xv$, where $x \in \Sigma$ and $v \in \Sigma^*$. By its construction, we have that $(r, P, c)(x, [xf \xrightarrow{v} wf], wf) = (s, Q, d)$. We could in a similar manner construct an element by which we could right multiply (s, Q, d) to obtain (r, P, c) . Thus $(r, P, c)\mathcal{R}(s, Q, d)$.

Finally, note that if $(r, P, c)\mathcal{R}(s, Q, d)$, the assumption that c and d lie on a cycle in P also implies that $c \mathcal{R} d$ in S . ■

Since P is a finite subdigraph, we have the following.

Corollary 6.1.2. *The \mathcal{R} -classes of $\mathcal{M}(S; \Sigma)$ are finite.*

Corollary 6.1.2 provides an alternative proof of Theorem 5.5.1(b), since all subgroups

are contained in \mathcal{R} -classes.

6.2 \mathcal{L} -Classes

\mathcal{L} -classes have a more complicated description than \mathcal{R} -classes, because of the different effects of multiplying on the right versus multiplying on the left. When (r, P, c) is multiplied by an element on the right, the root rf is fixed and P is contained in the new subdigraph. In contrast, when (r, P, c) is multiplied by an element on the left, the product can have a different root, an r -labeled edge is possibly added, and the subdigraph P is translated. Moreover, upon translation, the vertices and edges of P may collapse. As a consequence, \mathcal{L} -equivalent elements may have different roots and non-isomorphic subdigraphs; still, their digraph structures are related. Describing this relationship, and thereby characterizing the \mathcal{L} -classes, is the goal of Theorem 6.2.2. In order to prove this theorem, we need a few definitions and results about digraphs.

We will utilize the right semigroup action of Σ^* on S , and hence on the vertices and edges of the Cayley digraph $\text{Cay}(S; \Sigma)$. We review the notation for this action. If $c \in S$ and $w \in \Sigma^*$, then

$$c \cdot w = \begin{cases} c & \text{if } w = \epsilon \\ c(wf) & \text{if } w \neq \epsilon \end{cases}$$

In the Cayley digraph, the vertex obtained when c is acted upon by w is the vertex which is reached by starting at c and reading the word w . Similarly, if the edge $(c, r) \in E(\text{Cay}(S; \Sigma))$, then $(c, r) \cdot w = (c \cdot w, r) = (c(wf), r)$. We will show that this action is preserved by label-preserving digraph morphisms.

Lemma 6.2.1. *Let (S, Σ, f) be a semigroup system, $v \in S$, and let $\theta : \text{Cay}(S; \Sigma)_v^\dagger \rightarrow \text{Cay}(S; \Sigma)$ be a label-preserving digraph morphism. Furthermore, let $c \in V(\text{Cay}(S; \Sigma)_v^\dagger)$ and $w \in \Sigma^*$. Then*

- (a) $(c \cdot w)\theta = c\theta \cdot w$;
- (b) if $(c \cdot w, r) \in E(\text{Cay}(S; \Sigma)_v^\uparrow)$, then $(c \cdot w, r)\theta = (c\theta \cdot w, r)$;
- (c) if $P \subseteq \text{Cay}(S, \Sigma)$ is a rooted digraph, then $(cP)\theta = (c\theta)P$.

Proof: If $w = \epsilon$, then $(c \cdot w)\theta = c\theta = c\theta \cdot w$. For $|w| \geq 1$, we use induction on the length of w . If $|w| = 1$, then $(c, w) \in E(\text{Cay}(S; \Sigma))$. Since θ is a label-preserving digraph morphism and there is exactly one w -labeled edge leaving the vertex $c\theta$, we know that $(c, w)\theta = (c\theta, w)$. Thus $(c(wf))\theta = c\theta(wf)$, which can be rewritten using action notation as $(c \cdot w)\theta = c\theta \cdot w$.

Now, suppose $|w| = n$ for some $n \in \mathbb{N}$. Assume for all $v \in \Sigma^*$ with $|v| < n$ that $(c \cdot v)\theta = c\theta \cdot v$. We write w as $w = us$, where $u \in \Sigma^*$ and $s \in \Sigma$. Thus, since $|u| < n$ and $|s| = 1$,

$$\begin{aligned}
(c \cdot w)\theta &= ((c \cdot u) \cdot s)\theta \\
&= (c \cdot u)\theta \cdot s \\
&= (c\theta \cdot u) \cdot s \\
&= c\theta \cdot w.
\end{aligned}$$

Part (b) is a consequence of (a) and the fact that θ preserves edge labels. Turning to part (c), suppose P consists solely of the vertex \bullet_d . There exists some $w \in \Lambda^+$ such

that $wf = d$. Thus using the result from (a),

$$\begin{aligned}
(cP)\theta &= (c\{\bullet_d\})\theta \\
&= (\{\bullet_{cd}\})\theta \\
&= (\{\bullet_{c \cdot w}\})\theta \\
&= \{\bullet_{(c\theta) \cdot w}\} \\
&= \{\bullet_{(c\theta)d}\} \\
&= (c\theta)\{\bullet_d\} \\
&= (c\theta)P.
\end{aligned}$$

On the other hand, if P contains any edges, it cannot contain isolated vertices. Thus it is sufficient in this case to show that $E((cP)\theta) = E((c\theta)P)$. To this end, let $(x, r) \in E((cP)\theta)$. Then there exists some $(x', r) \in E(cP)$ such that $x'\theta = x$. Furthermore, there exists some $(x'', r) \in E(P)$ such that $cx'' = x'$. Let $w \in \Sigma^*$ be such that $wf = x''$. Hence from part (a), we have

$$\begin{aligned}
(x, r) &= (x'\theta, r) \\
&= ((cx'')\theta, r) \\
&= ((c(wf)\theta, r) \\
&= ((c \cdot w)\theta, r) \\
&= ((c\theta) \cdot w, r) \\
&= ((c\theta)(wf), r) \\
&= ((c\theta)x'', r) \\
&= (c\theta)(x'', r).
\end{aligned}$$

This implies that $(x, r) \in E((c\theta)P)$. The reverse inclusion can be proved by reversing the order of the steps. ■

We can now describe the \mathcal{L} -classes.

Theorem 6.2.2. *Assume $(r, P, c) \neq (s, Q, d)$. Then $(r, P, c)\mathcal{L}(s, Q, d)$ if and only if there exist elements $a, b \in S$ that satisfy the following:*

- (a) $ac = c$ and $bd = d$;
- (b) $aP_r^1 \subseteq P$ and $bQ_s^1 \subseteq Q$;
- (c) aP_r^1 and bQ_s^1 are isomorphic as labeled subdigraphs and there exists a labeled digraph isomorphism $aP_r^1 \rightarrow bQ_s^1$ that maps c to d .

Proof: Suppose $(r, P, c)\mathcal{L}(s, Q, d)$. Then there exist some (r, A, x) and (s, B, y) such that:

$$(r, P, c) = (r, A, x)(s, Q, d) = (r, A \cup xQ_s^1, xd) \quad (6.2.1)$$

$$(s, Q, d) = (s, B, y)(r, P, c) = (s, B \cup yP_r^1, yc). \quad (6.2.2)$$

The above equations show that $xQ_s^1 \subseteq P$ and $yP_r^1 \subseteq Q$. Combining these containment relationships, we have that

$$x(yP_r^1) \subseteq xQ \subseteq xQ_s^1 \subseteq P.$$

Using the same strategy,

$$xy(xQ_s^1) \subseteq xyP \subseteq xyP_r^1 \subseteq P.$$

Repeating this, we find that for all $i \in \mathbb{N}$ the following are true:

$$(xy)^i P_r^1 \subseteq P \quad (6.2.3)$$

$$x(yx)^i Q_s^1 \subseteq P \quad (6.2.4)$$

$$(yx)^i Q_s^1 \subseteq Q \quad (6.2.5)$$

$$y(xy)^i P_r^1 \subseteq Q. \quad (6.2.6)$$

Since the digraph P is finite, the element xy is periodic. Assume $k, m \in \mathbb{N}$ are the smallest values such that $(xy)^k = (xy)^{k+m}$. Note that this implies the equation $(yx)^{k+1} = (yx)^{k+m+1}$. We claim that $(xy)^k$ and $(yx)^{k+1}$ satisfy the requirements of a and b in conditions (a), (b), and (c). First, observe from Equations 6.2.1 and 6.2.2 that $c = xd$ and $d = yc$. By substituting each of these equations into the other, we obtain $c = xyc$ and $d = yxd$; i.e. xy and yx are left identities for c and d respectively. Hence $(xy)^k$ and $(yx)^{k+1}$ are also respective left identities. Second, Equations 6.2.3 and 6.2.5 indicate that part (b) is satisfied. It remains to find an isomorphism between $(xy)^k P_r^1$ and $(yx)^{k+1} Q_s^1$. To this end, we show that if we translate $(xy)^k P_r^1$ by y , the graph obtained is $(yx)^{k+1} Q_s^1$. First,

$$\begin{aligned} (yx)^{k+1} Q_s^1 &= y(xy)^k x Q_s^1 \\ &\subseteq y(xy)^k P \\ &\subseteq y(xy)^k P_r^1. \end{aligned}$$

Now, using the periodicity of xy and Equation 6.2.6,

$$\begin{aligned}
y(xy)^k P_r^1 &= y(xy)^{k+m} P_r^1 \\
&= (yx)^{k+1} y(xy)^{m-1} P_r^1 \\
&\subseteq (yx)^{k+1} Q \\
&\subseteq (yx)^{k+1} Q_s^1.
\end{aligned}$$

Thus $y(xy)^k P_r^1 = (yx)^{k+1} Q_s^1$. Similarly, we could show that translating $(yx)^{k+1} Q_s^1$ by x produces $(xy)^k P_r^1$. Because the digraphs $(xy)^k P_r^1$ and $(yx)^{k+1} Q_s^1$ are both finite, if translation by y collapsed any edges or vertices of $(xy)^k P_r^1$, then the digraph $(yx)^{k+1} Q_s^1$ would have fewer edges and vertices than $(xy)^k P_r^1$ and thus could not have $(xy)^k P_r^1$ as an image (under translation by x). Thus translation by y corresponds to a label-preserving digraph isomorphism, which we denote by $\theta : (xy)^k P_r^1 \rightarrow (yx)^{k+1} Q_s^1$. Notice that $c\theta = yc = d$. Thus we have satisfied the requirements of (c).

We now prove the converse. Suppose there exist some $a, b \in S$ that satisfy the three conditions. Moreover, let $\theta : aP_r^1 \rightarrow bQ_s^1$ be the label-preserving digraph isomorphism guaranteed by (c). Let $w \in \Sigma^*$ be a word such that $wf = c$. First we note using Lemma 6.2.1(a) that,

$$(a\theta)c = a\theta(wf) = (a\theta) \cdot w = (a \cdot w)\theta = (a(wf))\theta = (ac)\theta = c\theta = d. \quad (6.2.7)$$

Also, from conditions (b) and (c) of Lemma 6.2.1,

$$(a\theta)P_r^1 = (aP_r^1)\theta = bQ_s^1 \subseteq Q. \quad (6.2.8)$$

Since $(s, Q, d) \in \mathcal{M}(S, \Sigma)$, the digraph Q is sf -rooted. Moreover, Equation 6.2.8 implies that $a\theta \in Q$. Then $(s, Q, a\theta) \in \mathcal{M}(S, \Sigma)$. Combining the information in

Equations 6.2.8 and 6.2.7, we have that:

$$\begin{aligned} (s, Q, a\theta)(r, P, c) &= (s, Q \cup (a\theta)P_r^1, (a\theta)c) \\ &= (s, Q, d). \end{aligned}$$

Since θ is an isomorphism, θ^{-1} exists, it can be easily shown that $(r, P, b\theta^{-1})$, and $(r, P, b\theta^{-1})(s, Q, d) = (r, P, c)$. We conclude that $(r, P, c)\mathcal{L}(s, Q, d)$. \blacksquare

6.3 Descriptions of \mathcal{L} -Classes for Specific Classes of Semigroups

The structure of \mathcal{L} -related elements described in Theorem 6.2.2 can often be refined if we know more information about the semigroup.

Semigroup Systems of Groups

First we consider the \mathcal{L} -classes of graph expansions of semigroup systems of groups. We will show that \mathcal{L} -related elements have almost the same description as in the group graph expansion case. To this end, we will strengthen Lemma 6.0.5 by restricting to groups.

Corollary 6.3.1. *Let (S, Σ, f) be a semigroup system of a group S . Let $P \subseteq \text{Cay}(S; \Sigma)$ be a rooted subdigraph of $\text{Cay}(S; \Sigma)$ and let $c \in V(P)$, $d \in S$. If $\theta : P \rightarrow \text{Cay}(S; \Sigma)$ is a label-preserving digraph map which sends c to d , then the map θ is the same as the map corresponding to translation by dc^{-1} . Moreover, θ is the unique label-preserving digraph map sending c to d .*

Proof: We shall construct a larger semigroup system for S to which Lemma 6.0.5 applies. To this end, let $\bar{\Sigma}$ be a set of formal inverses for Σ . Consider the set $\Delta = \Sigma \cup \bar{\Sigma}$. We extend f to Δ by mapping elements of the form \bar{r} to $(rf)^{-1}$.

Let $Q \subseteq \text{Cay}(S; \Delta)$ be the digraph defined by the following: $V(Q) = V(P)$ and $E(Q) = E(P) \cup \{(x, \bar{r}) \mid (x(rf)^{-1}, r) \in E(P)\}$. In words, we obtain Q from P by inserting an inverse edge for every edge in P . Since P is a rooted subdigraph of $\text{Cay}(S; \Sigma)$, Q is a strongly connected subdigraph of $\text{Cay}(S; \Delta)$. In particular, c is a root of Q .

We extend θ to a map from Q as follows: let $\bar{\theta} : Q \rightarrow \text{Cay}(S; \Delta)$ send edges of the form (x, \bar{r}) to $(x\theta, \bar{r})$. The map $\bar{\theta}$ inherits the label-preserving property from θ . Clearly $c\bar{\theta} = d$. Because S is a group, we have $(dc^{-1})c = d$. Thus Lemma 6.0.5 indicates that $\bar{\theta}$ is the same as the map corresponding to translation by dc^{-1} . Moreover, as dc^{-1} is the unique element with which to left multiply c and obtain d , this implies that $\bar{\theta}$ is the unique label-preserving digraph map sending c to d . Finally, restricting to P , we see that θ is given by the map corresponding to translation by dc^{-1} and that θ is the unique label-preserving digraph map sending c to d . \blacksquare

We now characterize the \mathcal{L} -classes of graph expansions for the group case.

Corollary 6.3.2. *Let (S, Σ, f) be a semigroup system of a group S . Suppose $(r, P, c) \neq (s, Q, d)$. Then $(r, P, c)\mathcal{L}(s, Q, d)$ if and only if the following hold:*

$$(a') \quad P_r^1 = P \text{ and } Q_s^1 = Q;$$

$$(b') \quad c^{-1}P = d^{-1}Q.$$

Proof: Assume $(r, P, c)\mathcal{L}(s, Q, d)$. From Theorem 6.2.2(a), there exists some $a \in S$ such that $ac = c$. Since S is a group, $a = 1$, where 1 is the group identity of S . Thus for all $s \in S$, we have that $1s = s$, from which it follows that $1P_r^1 = P_r^1$. Appealing to Theorem 6.2.2(b) gives $1P_r^1 \subseteq P$. Noting that $P \subseteq P_r^1$ is always true, we conclude $P_r^1 = P$. Similarly $Q_s^1 = Q$.

From Theorem 6.2.2(c), there exists a label-preserving graph isomorphism θ from P_r^1 to Q_s^1 that maps c to d . From Corollary 6.3.1, the map θ corresponds to translation by dc^{-1} . Thus,

$$dc^{-1}P = dc^{-1}P_r^1 = (P_r^1)\theta = Q_s^1 = Q.$$

Multiplying both sides by d^{-1} yields $c^{-1}P = d^{-1}Q$.

Conversely, suppose (a') and (b'). We will show that conditions (a), (b), and (c) of Theorem 6.2.2 hold for the values $a = b = 1$, where 1 is the group identity of S . Clearly $1c = c$ and $1d = d$. Also, as observed above, $1P_r^1 \subseteq P$ and $1Q_s^1 \subseteq Q$. Finally, let $\theta : P_r^1 \rightarrow \text{Cay}(S; \Sigma)$ be the graph map that corresponds to translation by dc^{-1} . From (b'),

$$(P_r^1)\theta = dc^{-1}(P_r^1) = dd^{-1}Q_s^1 = Q_s^1.$$

Moreover, $c\theta = dc^{-1}c = d$. Thus condition (c) is satisfied. By Theorem 6.2.2, we conclude that $(r, P, c)\mathcal{L}(s, Q, d)$. ■

Left-zero Semigroups

We turn to left-zero semigroups. It will be useful to use the alternative description of the graph expansion given by Lemma 4.3.1: we replace the notation (r, P, c) by $(r, \Sigma(P))$, because, for left-zero semigroups, $rf = c$ and P is determined by its edge-label set $\Sigma(P)$.

Proposition 6.3.3. *Let (S, Σ, f) be a semigroup system of a left-zero semigroup S . Suppose $(r, \Sigma(P)) \neq (s, \Sigma(Q))$. Then $(r, \Sigma(P))\mathcal{L}(s, \Sigma(Q))$ if and only if the following hold:*

$$(a') \quad r, s \in \Sigma(P);$$

$$(b') \quad \Sigma(P) = \Sigma(Q).$$

Proof: Suppose $(r, \Sigma(P))\mathcal{L}(s, \Sigma(Q))$. Then there exists an element $(a, \Sigma(A))$ such that $(a, \Sigma(A))(r, \Sigma(P)) = (s, \Sigma(Q))$. Expanding the left side, we see that $(a, \Sigma(A) \cup \Sigma(P) \cup \{r\}) = (s, \Sigma(Q))$. Thus $\Sigma(P) \subseteq \Sigma(Q)$ and $r \in \Sigma(Q)$. Similarly, we could show that $\Sigma(Q) \subseteq \Sigma(P)$ and $s \in \Sigma(P)$. Thus $r, s \in \Sigma(P)$ and $\Sigma(P) = \Sigma(Q)$.

Conversely, suppose (a') and (b') hold. It is easy to see that

$$(r, \Sigma(P))(s, \Sigma(Q)) = (r, \Sigma(P) \cup \Sigma(Q) \cup \{s\}) = (r, \Sigma(P)).$$

Similarly $(s, \Sigma(Q))(r, \Sigma(P)) = (s, \Sigma(Q))$. Thus $(r, \Sigma(P))\mathcal{L}(s, \Sigma(Q))$. ■

Right-zero Semigroups

The next case concerns right-zero semigroups.

Corollary 6.3.4. *Let (S, Σ, f) be a semigroup system of a right-zero semigroup S . Suppose $(r, P, c) \neq (s, Q, d)$. Then $(r, P, c)\mathcal{L}(s, Q, d)$ if and only if the following hold:*

$$(a') \quad c = d;$$

$$(b') \quad P = Q;$$

$$(c') \quad P \text{ contains both an } r\text{-labeled and an } s\text{-labeled edge.}$$

Proof: One observations about $\text{Cay}(S; \Sigma)$ will be very useful. For every $t \in \Sigma$, if $c = tf$, then c is the only vertex with edges labeled by t entering it. This implies that if $P \subseteq \text{Cay}(S; \Sigma)$ is a digraph such that every vertex of P has an edge entering it, then the only label-preserving digraph morphism $P \rightarrow \text{Cay}(S; \Sigma)$ is the identity map.

Suppose $(r, P, c)\mathcal{L}(s, Q, d)$. From Theorem 6.2.2(b), there exist some $a, b \in S$ such that $aP_r^1 \subseteq P$ and $bQ_s^1 \subseteq Q$. However, since S is a right-zero semigroup, $P \subseteq aP_r^1$ and $Q \subseteq bQ_s^1$. Thus $P = aP_r^1$ and $Q = bQ_s^1$. From Theorem 6.2.2(c), $P = aP_r^1$ and $Q = bQ_s^1$ are isomorphic as labeled graphs. Noting that since $(a, r) \in E(P)$, every vertex of P has an edge entering it. Thus from our earlier observation, the only label-preserving isomorphism possible between P and Q is the identity map. Thus $P = Q$. Moreover, $c = d$. Finally, since $(a, r), (b, s) \in E(P)$, it is clear that P contains both an r -labeled and an s -labeled edge.

Conversely, suppose conditions (a'), (b'), and (c'). By (c'), we are guaranteed that there are elements $a, b \in S$, such that $(a, r), (b, s) \in E(P)$. Since S is a right-zero

semigroup, this implies that conditions (a) and (b) of Theorem 6.2.2 hold for a and b . Moreover, the identity map from P to itself satisfies conditions (c). Thus, applying Theorem 6.2.2 shows that $(r, P, c)\mathcal{L}(s, Q, d)$. ■

Semilattices

Semilattices are the fourth case that we will look at.

Corollary 6.3.5. *Let (S, Σ, f) be a semigroup system of a semilattice S . Suppose $(r, P, c) \neq (s, Q, d)$. Then $(r, P, c)\mathcal{L}(s, Q, d)$ if and only if the following hold:*

$$(d') \quad c = d;$$

(b') *there exists an element $a \in S$, with $c \leq a$, such that (r, P, a) and (s, Q, a) are idempotents and $aP_r^1 = aQ_s^1$.*

Proof: Suppose $(r, P, c)\mathcal{L}(s, Q, d)$. This implies that $c\mathcal{L}d$ in S . However, as S is a semilattice, it follows that $c = d$. From Theorem 6.2.2(a), there exist some $a, b \in S$ such that $ac = c$, $bc = c$. We wish to show that $a = b$. Choose $u \in \Sigma^*$ such that $rf \xrightarrow{u} a$ is a path in P . It follows that $a = (ru)f$. Write u as $u = u_1 \dots u_n$ where each $u_i \in \Sigma$. We note that $a \leq rf$ and $a \leq u_i f$ for each letter u_i . Thus, the digraph aP_r^1 contains a loop at a labeled by r , and a loop at a labeled by u_i for each u_i . From Theorem 6.2.2(c), we know that aP_r^1 and bQ_s^1 are isomorphic as labeled digraphs. We observe that a rooted subdigraph of the Cayley digraph of a semilattice has a unique root. Thus, this isomorphism must map the root of aP_r^1 to the root of bQ_s^1 , i.e. a to b . This then implies that the digraph bQ_s^1 contains a loop at b labeled by r , and a loop at b labeled by u_i for each u_i . Thus $b \leq (ru)f = a$. Similarly we could show that $a \leq b$. We conclude that $a = b$.

In the Cayley digraph of a semilattice, isomorphic subdigraphs with the same root must be the same subdigraph. Combining the fact that $a = b$ with Theorem 6.2.2(c), we know that $aP_r^1 = bQ_s^1$. From Theorem 6.2.2(b), we know that $aP_r^1 \subseteq P$

and $Q_s^1 \subseteq Q$. Thus by Proposition 4.6.1(a), we have that (r, P, a) and (s, Q, a) are idempotents.

Conversely, suppose conditions (a') and (b'). Simple observation shows that these imply conditions (a), (b), and (c) of Theorem 6.2.2. Thus by Theorem 6.2.2, we conclude that $(r, P, c)\mathcal{L}(s, Q, d)$. ■

Once we have covered the remaining Green's relations, we will describe the egg-box diagram for a semilattice with two generators, using Corollary 6.3.5 to determine the \mathcal{L} -classes.

6.4 \mathcal{H} -Classes

We obtain a characterization of \mathcal{H} -classes by combining the requirements of \mathcal{R} -classes with slightly modified \mathcal{L} -class requirements:

Theorem 6.4.1. *Assume $(r, P, c) \neq (s, Q, d)$. Then $(r, P, c)\mathcal{H}(s, Q, d)$ if and only if the following hold:*

- (a) $r = s$, $P = Q$, and c and d are on a cycle in P ;
- (b) there exist some $a, b \in S$ such that $ac = c$, $bd = d$, $aP_r^1, bP_r^1 \subseteq P$, and there exists a label-preserving digraph isomorphism $aP_r^1 \rightarrow bP_r^1$ that maps c to d .

Proof: Let $(r, P, c)\mathcal{H}(s, Q, d)$. Then we know $(r, P, c)\mathcal{R}(s, Q, d)$ and by Theorem 6.1.1 we are guaranteed condition (a). We also know that $(r, P, c)\mathcal{L}(s, Q, d)$. Since $P = Q$, Theorem 6.2.2 gives condition (b).

Conversely, if we assume the conditions (a) and (b), then we can use Theorems 6.1.1 and 6.2.2 to determine that (r, P, c) and (s, Q, d) are \mathcal{R} - and \mathcal{L} -related. Thus they are \mathcal{H} -related. ■

Given the number of conditions, the typical element of a semigroup graph expansion is \mathcal{H} -related only to itself. However, if an element (r, P, c) is \mathcal{H} -related to a different element, then Theorem 6.4.1 guarantees the existence of a subdigraph of P with nice structure. We describe this in the following Lemma.

Lemma 6.4.2. *Suppose $(r, P, c)\mathcal{H}(r, P, d)$. Then it follows that $P_c^\uparrow = P_d^\uparrow$. Additionally, there exists a label-preserving digraph automorphism $P_c^\uparrow \rightarrow P_d^\uparrow$ that sends c to d .*

Proof: Since (r, P, c) and (r, P, d) are \mathcal{H} -related, from Theorem 6.4.1 (a) we know there is a cycle in P containing c and d . It follows that $P_c^\uparrow = P_d^\uparrow$.

From Theorem 6.4.1 (b), there exist some $a, b \in S$ such that $ac = c$, $bd = d$, and there exists a label-preserving digraph isomorphism $\theta : aP_r^1 \rightarrow bP_r^1$ that maps c to d . We wish to show that by restricting θ to P_c^\uparrow , we obtain the desired automorphism. Because $aP_c^\uparrow \subseteq aP_r^1$, we can clearly restrict θ to aP_c^\uparrow . However, since a is a left identity for c , it is also a left identity for the graph accessible from c , i.e. $aP_c^\uparrow = P_c^\uparrow$. Moreover, since $c\theta = d$ and P_c^\uparrow is by definition c -rooted, $P_c^\uparrow\theta \subseteq P_d^\uparrow = P_c^\uparrow$. However, as θ is an isomorphism and P_c^\uparrow is finite, this is only possible if $P_c^\uparrow\theta = P_c^\uparrow$. We conclude that θ is the desired label-preserving automorphism of P_c^\uparrow . ■

Even more can be said about the structure of an element (r, P, c) if $H_{(r, P, c)}$ is a subgroup of $\mathcal{M}(S; \Sigma)$. Before stating the result, we review some notation. To every labeled graph Γ we can associate a label-preserving automorphism group, which we denote by $\text{Aut}(\Gamma)$. Suppose $\Gamma \subseteq \text{Cay}(S; \Sigma)$ is an x -rooted subdigraph and $\theta \in \text{Aut}(\Gamma)$. There exists some $a \in S$ such that $ax = x\theta$. By Lemma 6.0.5, the map θ is uniquely given by translation by a . Thus, we will write θ_a for the automorphism that corresponds to translation by a . Moreover, since translation by a followed by translation by b is the same as translation by ab , we have that $\theta_a\theta_b = \theta_{ab}$.

Theorem 6.4.3. *If $H_{(r, P, c)}$ is a subgroup of $\mathcal{M}(S; \Sigma)$, then $H_{(r, P, c)}$ is isomorphic to the (label-preserving) automorphism group of P_c^\uparrow .*

Proof: Since $H_{(r, P, c)}$ is a subgroup, there is an element (r, P, e) which is the identity

of the subgroup. The following fact about e will be useful: left translation of P_e^\uparrow by e fixes P_e^\uparrow . To see why, let $d \in V(P_e^\uparrow)$. From the definition of P_e^\uparrow , there exists some word $w \in \Sigma^*$ labeling an $e \rightarrow d$ path in P and therefore $e(wf) = d$. Since $e^2 = e$, it follows that $ed = d$. Thus, as left multiplication by e fixes vertices, we have that $eP_e^\uparrow = P_e^\uparrow$.

Let $(r, P, x) \in H_{(r, P, c)}$. Lemma 6.4.2 ensures that θ_x is an automorphism of P_c^\uparrow . Therefore we can define the following map:

$$\beta : H_{(r, P, c)} \rightarrow \text{Aut}(P_c^\uparrow) \quad \text{by} \quad (r, P, x) \mapsto \theta_x.$$

We show that β is a homomorphism. Suppose $(r, P, a), (r, P, b) \in H_{(r, P, c)}$. Observe that, as (r, P, e) is the identity, $(r, P \cup aP_r^1, ae) = (r, P, a)(r, P, e) = (r, P, a)$, whereupon we see that $aP_r^1 \subseteq P$. Using this we have:

$$\begin{aligned} (r, P, a)\beta(r, P, b)\beta &= \theta_a \circ \theta_b \\ &= \theta_{ab} \\ &= (r, P, ab)\beta \\ &= (r, P \cup aP_r^1, ab)\beta \\ &= ((r, P, a)(r, P, b))\beta. \end{aligned}$$

Suppose $(r, P, a)\beta = (r, P, b)\beta$. Then $\theta_a = \theta_b$, whereupon it follows from Lemma 6.3.1 that $a = b$. We conclude that β is injective.

We now show that it is surjective. Let $\theta_x \in \text{Aut}(P_c^\uparrow)$. Then

$$x = xe \in xP_c^\uparrow = P_c^\uparrow \theta_x = P_c^\uparrow \subseteq P. \quad (6.4.1)$$

Thus $(r, P, x) \in \mathcal{M}(S; X)$. We want to show that $(r, P, x) \in H_{(r, P, c)}$. Our strategy will be to show that $(r, P, x)\mathcal{H}(r, P, e)$. From Equation 6.4.1, we see that $x \in P_c^\uparrow$. Combining this with Lemma 6.4.2 implies that $x \in P_e^\uparrow$. By its definition, every vertex

of P_e^\uparrow is accessible from e . This implies that every vertex of P_e^\uparrow is also accessible from $e\theta_x = xe = x$. We conclude that e and x are on a cycle in P . Thus we have Theorem 6.4.1(a).

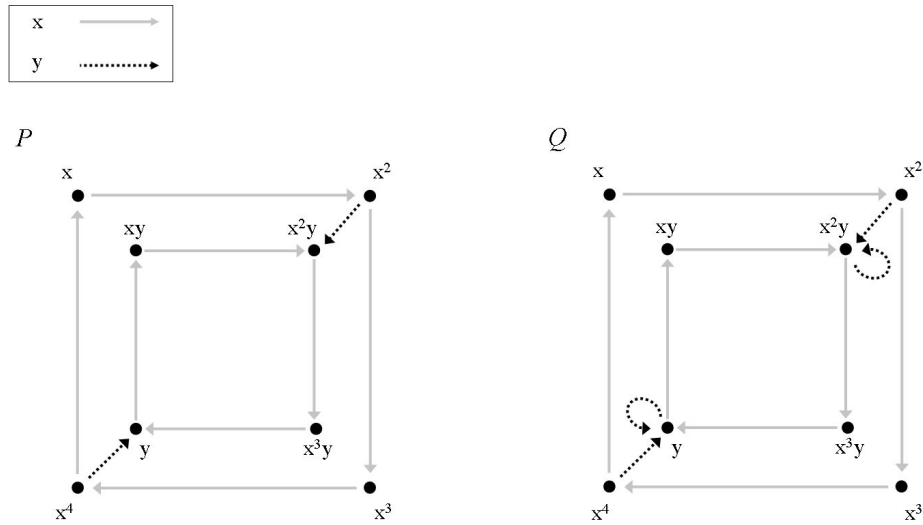
We now want to show that e satisfies the role of both a , b referred to in Theorem 6.4.1 (b). Being the local identity, $e^2 = e$ and $ex = x$. Since (r, P, e) is idempotent, we have that $(r, P, e) = (r, P, e)(r, P, e) = (r, P \cup eP_r^1, e)$, from which it follows that $eP_r^1 \subseteq P$. Moreover, this shows that

$$P_e^\uparrow = eP_e^\uparrow \subseteq eP_r^1 \subseteq P_e^\uparrow.$$

Thus $eP_r^1 = P_e^\uparrow$, whereupon θ_x is a label-preserving automorphism of eP_r^1 that sends e to x . By Theorem 6.4.1, $(r, P, e)\mathcal{H}(r, P, x)$. Furthermore $(r, P, x)\beta = \theta_x$, showing that β is surjective. We conclude that β is a group isomorphism. ■

We give an example of two \mathcal{H} -classes, one that is a subgroup and one that is not.

Example: Let S be the direct product of the cyclic group $C_4 = \{e, c, c^2, c^3\}$ (we denote the identity by e) and the trivial group with zero $Y = \{1, 0\}$. Let $x = (c, 1)$ and $y = (e, 0)$. Since these two elements generate S as a semigroup, we can form the semigroup system $(S, \{x, y\}, id)$. Let P and Q be the subdigraphs shown below.



We consider two \mathcal{H} -classes: $H_{(x,P,x^2y)}$ and $H_{(x,Q,x^2y)}$.

- $H_{(x,P,x^2y)}$ contains two elements, (x, P, x^2y) and (x, P, y) . (If we insert the value x^4 for a and b in Theorem 6.4.1, it is easy to see that $(x, P, y) \mathcal{H} (x, P, x^2y)$. With a bit more work, any other \mathcal{H} -related elements can be ruled out.) It is not a subgroup, because neither of its elements is idempotent.
- $H_{(x,Q,x^2y)}$ also contains two elements, (x, Q, x^2y) and (x, Q, y) . (Again, this can be obtained by using the values x^4 or y for a and b in Theorem 6.4.1. As before, any other \mathcal{H} -related elements can be ruled out.) The class $H_{(x,Q,x^2y)}$ is a subgroup. Its identity is (x, Q, y) . It is isomorphic to C_2 , which is clearly the automorphism group of Q_y^\uparrow .

■

6.5 \mathcal{D} - and \mathcal{J} -Classes

We would like to end our investigation of Green's relations with a description of the \mathcal{D} - and \mathcal{J} -classes. Recall that for a semigroup S , $x\mathcal{D}y$ if and only if we can find some $c \in S$, such that $x\mathcal{R}c$ and $y\mathcal{L}c$, and $x\mathcal{J}y$ if and only if $S^1xS^1 = S^1yS^1$. Our first task will be to characterize the \mathcal{D} -classes as we did for the \mathcal{R} -, \mathcal{L} -, and \mathcal{H} -classes.

Theorem 6.5.1. *Assume $(r, P, c) \neq (s, Q, d)$. Then $(r, P, c)\mathcal{D}(s, Q, d)$ if and only if there exist elements $a, b \in S$ that obey the following:*

- (a) $ac = c$ and $bd = d$;
- (b) $aP_r^1 \subseteq P$ and $bQ_s^1 \subseteq Q$;
- (c) aP_r^1 and bQ_s^1 are isomorphic as labeled digraphs and there exists a label-preserving isomorphism $\theta : aP_r^1 \rightarrow bQ_s^1$ such that subdigraph bQ_s^1 contains a cycle connecting $c\theta$ and d .

Proof: Suppose $(r, P, c)\mathcal{D}(s, Q, d)$ and assume $(r, P, c) \neq (s, Q, d)$. We wish to show parts (a) and (b). There exists an element $(t, A, x) \in \mathcal{M}(S, \Sigma)$ such that

$(r, P, c)\mathcal{R}(t, A, x)$ and $(t, A, x)\mathcal{L}(s, Q, d)$. From Proposition 6.1.1, we know $t = r$ and $A = P$, and thus we replace t by r and A by P for the rest of this proof. Since $(r, P, x)\mathcal{L}(s, Q, d)$, by Proposition 6.2.2(a) and (b) there exists an $a, b \in S$ such that $ax = x$, $bd = d$, $aP_r^1 \subseteq P$, and $bQ_s^1 \subseteq Q$. Because $x\mathcal{R}c$, there exists a $y \in S$ such that $xy = c$. Thus $ac = axy = xy = c$. Thus we have (a) and (b).

Finally we prove (c). Since left multiplication by a fixes x , it also fixes every vertex accessible from x . By Proposition 6.1.1, we know there is a cycle C in P containing x and c . Combining this with the previous fact, the cycle C is contained in aP_r^1 . Proposition 6.2.2(c) states that aP_r^1 and bQ_s^1 are isomorphic as labeled subdigraphs and there exists a labeled digraph isomorphism $\theta : aP_r^1 \rightarrow bQ_s^1$ that maps x to d . The function θ also maps the cycle C to a cycle in bQ_s^1 ; the latter contains the vertices $x\theta = d$ and $c\theta$. Thus we have obtained (c).

Conversely, let $(r, P, c), (s, Q, d) \in \mathcal{M}$ and $a, b \in S$ satisfy (a), (b), and (c). From (c), we know there is a cycle in bQ_s^1 containing x and $c\theta$. Thus we can find words $v, w \in \Sigma^*$ such that Q contains the paths $c\theta \xrightarrow{v} x$ and $x \xrightarrow{w} c\theta$. Since θ is an isomorphism mapping aP_r^1 to bQ_s^1 , the existence of a cycle $c\theta \xrightarrow{vw} c\theta$ in bQ_s^1 implies that there exists a cycle $c \xrightarrow{vw} c$ in aP_r^1 and hence in P . This cycle contains the vertices c and $c \cdot v$. Thus appealing to Proposition 6.1.1, we have $(r, P, c)\mathcal{R}(r, P, c \cdot v)$.

Moreover, using condition (a), we see that $a(c \cdot v) = (ac) \cdot v = c \cdot v$. Combining this with the second half of (a), we satisfy Proposition 6.2.2 (a). Proposition 6.2.2 (b) follows from (b). Finally, from Lemma 6.2.1(a), we have that $(c \cdot v)\theta = (c\theta) \cdot v = x$. Thus we obtain Proposition 6.2.2(c) as well, whereupon we have that $(r, P, c \cdot v)\mathcal{L}(s, Q, d)$. We conclude that $(r, P, c)\mathcal{D}(s, Q, d)$. ■

We now wish to show that $\mathcal{D} = \mathcal{J}$ for semigroup graph expansions. Our approach will be constructive and rely on the structure of graph expansion elements.

Theorem 6.5.2. *Let (S, Σ, f) be a semigroup system. Then in $\mathcal{M}(S; \Sigma)$, $\mathcal{D} = \mathcal{J}$.*

Proof: It is a basic result of semigroup theory that $\mathcal{D} \subseteq \mathcal{J}$. We wish to show the reverse containment. Suppose for some $(r, P, c) \neq (s, Q, d)$ that $(r, P, c)\mathcal{J}(s, Q, d)$.

Then there exist elements (r, P, x) , (s, Q, y) , (t, C, w) , and (u, D, z) such that:

$$(r, P, c) = (r, P, x)(s, Q, d)(t, C, w) \quad (6.5.1)$$

$$(s, Q, d) = (s, Q, y)(r, P, c)(u, D, z). \quad (6.5.2)$$

By continually inserting these equations into each other, we obtain the following for any $i \in \mathbb{N}$:

$$(r, P, c) = ((r, P, x)(s, Q, y))^i (r, P, c) ((u, D, z)(t, C, w))^i \quad (6.5.3)$$

$$(s, Q, d) = (s, Q, y) ((r, P, x)(s, Q, y))^i (r, P, c) ((u, D, z)(t, C, w))^i (u, D, z). \quad (6.5.4)$$

Inspection of Equation 6.5.3 indicates that for all $i \in \mathbb{N}$, we have $(xy)^i \in V(P)$. (Alternatively, we could justify this observation using the same arguments as used in the proof for \mathcal{L} -classes.) Since P is a finite digraph, xy is periodic. Let k and m be the smallest natural numbers such that $(xy)^k = (xy)^{k+m}$. From Lemma 5.1.1,

$$((r, P, x)(s, Q, y))^{k+m} = ((r, P, x)(s, Q, y))^{k+2m}. \quad (6.5.5)$$

We wish to show that $(r, P, c)(u, D, z)\mathcal{R}(r, P, c)$. First, using Equations 6.5.3 and 6.5.5 we have that:

$$\begin{aligned} (r, P, c) &= ((r, P, x)(s, Q, y))^{k+2m} (r, P, c) ((u, D, z)(t, C, w))^{k+2m} \\ &= ((r, P, x)(s, Q, y))^{k+m} (r, P, c) ((u, D, z)(t, C, w))^{k+2m} \\ &= (r, P, c) ((u, D, z)(t, C, w))^m \\ &= ((r, P, c)(u, d, z)) ((t, C, w)((u, D, z)(t, C, w))^{m-1}). \end{aligned}$$

and obviously we can obtain $(r, P, c)(u, D, z)$ from (r, P, c) by multiplying the latter on the right by (u, D, z) . Thus $(r, P, c)(u, D, z)\mathcal{R}(r, P, c)$.

Now we wish to show that $(r, P, c)(u, D, z)\mathcal{L}(s, Q, d)$. In this case, we use Equations 6.5.2, 6.5.3, 6.5.4 and 6.5.5:

$$\begin{aligned}
(r, P, c)(u, D, z) &= ((r, P, x)(s, Q, y))^{k+m}(r, P, c)((u, D, z)(t, C, w))^{k+m}(u, D, z) \\
&= ((r, P, x)(s, Q, y))^{k+2m}(r, P, c)((u, D, z)(t, C, w))^{k+m}(u, D, z) \\
&= ((r, P, x)(s, Q, y))^{m-1}(r, P, x)(s, Q, y)((r, P, x)(s, Q, y))^{k+m}(r, P, c) \\
&\quad ((u, D, z)(t, C, w))^{k+m}(u, D, z) \\
&= (((r, P, x)(s, Q, y))^{m-1}(r, P, x))(s, Q, d).
\end{aligned}$$

Combining this result with Equation 6.5.2 shows that $(r, P, c)(u, D, z)\mathcal{L}(s, Q, d)$. We conclude that $(r, P, c)\mathcal{D}(s, Q, d)$. ■

We now consider certain finiteness properties of \mathcal{J} -classes. Like the \mathcal{L} -classes, \mathcal{J} -classes can be finite or infinite. We will soon give an example of a semigroup graph expansion that contains both types. Before doing that, we investigate the finite- \mathcal{J} -above property. Given two elements $x, y \in S$, we say that x is \mathcal{J} -above y ($x \geq_{\mathcal{J}} y$) if $S^1 y S^1 \subseteq S^1 x S^1$. This is equivalent to the existence of some $a, b \in S$ such that $y = axb$. A semigroup is called finite- \mathcal{J} -above, if for each $y \in S$, the set $\{x \in S \mid x \geq_{\mathcal{J}} y\}$ is finite. In the following proposition we show that if a semigroup is finite- \mathcal{J} -above, then its graph expansion (for any system) is as well. We note that Elston proves the same result for the semigroup Cayley expansion, but uses properties of derived categories to obtain it [3].

Proposition 6.5.3. *Let (S, Σ, f) be a semigroup system. If S is finite- \mathcal{J} -above and Σ is finite, then $\mathcal{M}(S; \Sigma)$ is finite- \mathcal{J} -above.*

Proof: Suppose S is finite- \mathcal{J} -above. Let $(r, P, c) \in \mathcal{M}(S; \Sigma)$. Consider the set

$$X = \{(s, Q, d) \mid (s, Q, d) \geq_{\mathcal{J}} (r, P, c)\}.$$

If X is finite, we are done. By way of contradiction, suppose X is infinite. We make two observations:

1. There are a finite number of elements of S that are chosen vertices for elements of X . This is because all these elements are \mathcal{J} -above c and we assumed that S is finite- \mathcal{J} -above.
2. All roots of elements of X come from a finite subset of Σ . This is because P is finite and if $(s, Q, d) \in X$ with $(s, Q, d) \neq (r, P, c)$, then P contains an s -labeled edge.

Combining these two observations with our assumption that X is infinite, we know there exists some $s \in \Sigma$ and $d \in S$ for which there are an infinite number of graphs Q_i such that $(s, Q_i, d) \in X$, $i \in \mathbb{N}$. We note that for each of these elements, there exists some $(r, A_i, a_i), (t_i, B_i, b_i)$ such that

$$(r, P, c) = (r, A_i, a_i)(s, Q_i, d)(t_i, B_i, b_i).$$

Note that for each i , we know that $a_i \in V(P)$. Since P is finite, there is some $a \in V(P)$ for which there are an infinite number of a_i with $a_i = a$. We thus specify a new subset:

$$X_a = \{(s, Q_i, d) \mid a = a_i\}.$$

Note that if $(s, Q_i, d) \in X_a$, then $a(Q_i)_s^1 \subseteq P$. Construct the graph

$$\Gamma = \bigcup_{(s, Q_i, d) \in X_a} Q_i.$$

We claim that $V(\Gamma)$ is infinite. To see this, observe that Γ is the union of an infinite number of distinct graphs. However, the edge label set of Γ is finite because it is contained in the edge label set of P . Moreover, since Γ is a subset of the Cayley digraph, each vertex has at most one edge of each label emerging from it. Thus $V(\Gamma)$ must be infinite. On the other hand, $a\Gamma_s^1 \subseteq P$. For each vertex $v \in V(P)$, we form

the set $Y_v = \{y \in V(\Gamma) \mid ay = v\}$. Since P is finite, there exists some v for which Y_v is infinite. However every $y \in Y_v$ is \mathcal{J} -above v since $ay1 = v$. This contradicts the fact that every element of S is finite- \mathcal{J} -above. Thus, the assumption that X is infinite is incorrect. We conclude that $\mathcal{M}(S; \Sigma)$ is finite- \mathcal{J} -above. ■

Not all graph expansions are finite- \mathcal{J} -above. Example 6.6 in the next section has two infinite \mathcal{J} -classes.

6.6 Examples with Eggbox Diagrams

We give two examples in this section. The first is a free semilattice on two generators. The second provides an example of a semigroup whose graph expansion has infinite \mathcal{L} - and \mathcal{J} -classes (and hence infinite \mathcal{D} -classes as well).

Example: Let S be the free semilattice on two generators,

$$S = \text{sgp}\langle x, y \mid x = x^2, y = y^2, xy = yx \rangle.$$

From this we form the related system, $(S, \{x, y\}, id)$. The Cayley digraph is shown below:

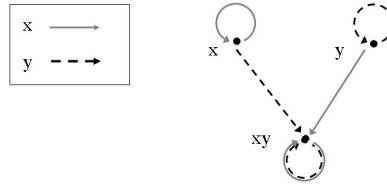


Figure 6.1: The Cayley digraph of the semigroup system $(S, \{x, y\}, id)$.

The graph expansion $\mathcal{M}(S; \{x, y\})$ contains 36 elements. To see how this number is derived, first note that the number of elements with x as the root is the same as the

number with y as the root. Thus, we need only count those of form (x, P, c) . Using the table below, the total number is $2 \times (16 + 2) = 36$.

Case	Possible edges, possible chosen vertices	Number of Elements
$(x, y) \notin E(P)$	(x, x) may be in $E(P)$; chosen vertex is x	2
$(x, y) \in E(P)$	$(x, x), (xy, x), (xy, y)$ may be in $E(P)$; chosen vertex: x or xy	$2^4 = 16$

Upon examining the Cayley digraph $\text{Cay}(S; \{x, y\})$, we see that there are no non-trivial cycles. Thus from Theorem 6.1.1, it follows that each of the 36 elements is in a one-element \mathcal{R} -class. There are 32 \mathcal{L} -classes with exactly one element and one \mathcal{L} -class with four elements. There are two reasons that the majority of elements are in single element \mathcal{L} -classes. First, any element of the form (x, P, x) is \mathcal{L} -related only to itself. To see why, suppose $(x, P, x)\mathcal{L}(t, Q, d)$. By Corollary 6.3.5(a'), $d = x$. Since x is only accessible from itself, $t = x$. Moreover, the only idempotent greater than or equal to x is itself. Thus by Corollary 6.3.5(b') (x, P, x) and (x, Q, x) are idempotents. Hence $P = xP_x^1$ and $Q = xQ_x^1$. Again using the corollary $P = xP_x^1 = xQ_x^1 = Q$.

The second reason is that any element of the form (x, P, xy) where P does not contain both the edges (xy, x) and (xy, y) is also \mathcal{L} -related only to itself. For this case, the reasons for this lie with Corollary 6.3.5(b'). This condition tells us that there exists some a such that (x, P, a) is idempotent. If $a = x$ and we suppose that $(x, P, x)\mathcal{L}(t, Q, d)$, then using the exact same reasoning as in the previous paragraph, it is easy to show that $t = x$, $P = Q$, and $d = x$. If $a = xy$, then Corollary 6.3.5(b') says that (x, P, xy) is idempotent. From Proposition 4.6.1(a), we have that $xyP_r^1 \subseteq P$, whereupon $(xy, x), (xy, y) \in E(P)$, a contradiction.

There are four elements with chosen vertex xy and digraphs containing the loops (xy, x) and (xy, y) . They are shown in Figure 6.2. They constitute the one nontrivial \mathcal{L} -class.

From Theorem 6.5.2 and this analysis of the \mathcal{R} - and \mathcal{L} -relations, it follows that

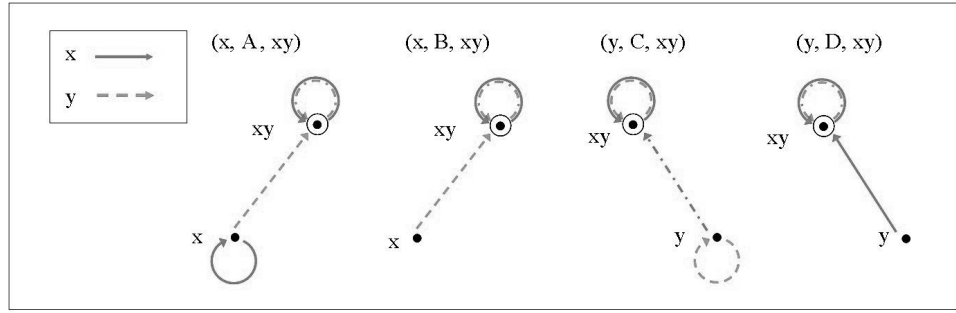


Figure 6.2: The four elements of $\mathcal{M}(S; \Sigma)$ that are in the same \mathcal{L} -class.

$\mathcal{R} = \mathcal{H}$ and $\mathcal{L} = \mathcal{D} = \mathcal{J}$. The \mathcal{H} -classes that are groups are those that consist of one idempotent. A partial eggbox diagram for $\mathcal{M}(S; \Sigma)$ is shown in Figure 6.3.

■

We give an example of a semigroup graph expansion with infinite \mathcal{L} -classes and hence also infinite $\mathcal{D} = \mathcal{J}$ -classes.

Example: Let $S = \text{sgp}\langle x, y \mid x = x^3, xy = x \rangle$. We use the semigroup system $(S, \{x, y\}, id)$. The Cayley digraph $\text{Cay}(S; \Sigma)$ is shown in Figure 6.4.

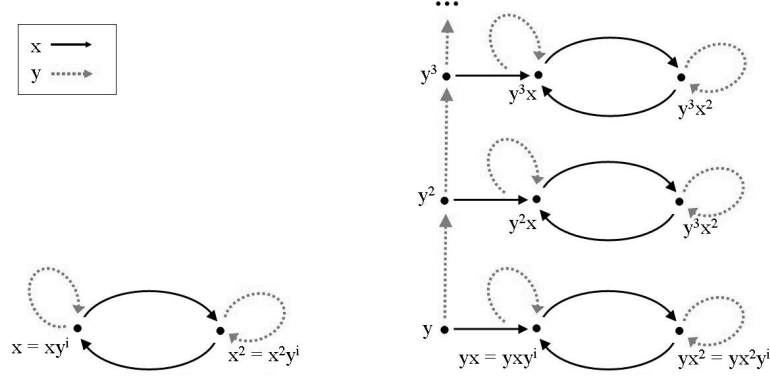


Figure 6.4: The Cayley digraph of the semigroup $S = \text{sgp}\langle x, y \mid x = x^3, xy = x \rangle$.

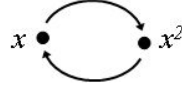
Inspecting the Cayley digraph in Figure 6.4, we see that an \mathcal{R} -class $R_{(r, P, c)}$ will contain two elements if there is a cycle of length two in P passing through c . Otherwise it will contain one element.

We can use Theorem 6.2.2 to describe the \mathcal{L} -classes of $\mathcal{M}(S, \Sigma, f)$. A few observations about S will help. First, note that all elements of S take one of the following normal forms: $\{x, x^2, y^i, y^i x, y^i x^2\}$, where $i \in \mathbb{N}$. We can then determine the left identities of the normal forms:

Left identity	Elements
x^2	x, x^2
$--$	y^i
$y^i x^2$	$y^i x, y^i x^2$

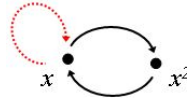
Suppose an element (x, P, x) is \mathcal{L} -related to another element. We can use Theorem 6.2.2 (a) and (b) to deduce information about the digraph P . First, the digraph P contains the left identity of x , namely x^2 . Thus (x, x) is an edge of P . Moreover, the

subdigraph containment $x^2P_x^1 \subseteq P$ implies that the edge (x^2, x) is also in P . If P contains no y -labeled loops, then P has the form



and (x, P, x) is not \mathcal{L} -related to any element of the form (y, Q, d) . The element (x, P, x) is \mathcal{L} -related to one other element, (x, P, x^2) . This \mathcal{L} -class is also an \mathcal{R} -, \mathcal{H} -, \mathcal{J} -, and \mathcal{D} -class.

We now describe what we know about the classes of (x, P, x) if P contains the y -labeled loop at x , but not the y -labeled loop at x^2 ; i.e. P has the form:



By Theorem 6.1.1, $(x, P, x)\mathcal{R}(x, P, x^2)$. However, (x, P, x) is not \mathcal{L} -related to (x, P, x^2) , because we can not satisfy Theorem 6.2.2(a) and (c); i.e. x^2 is the only possible value for a and b of Theorem 6.2.2 (a), but there is no digraph isomorphism from $x^2P_x^1 = P$ to itself that sends x to x^2 . Similarly, we could show that (x, P, x) is not \mathcal{L} -related to any other elements. Thus, (x, P, x) is a single element in an \mathcal{L} -, \mathcal{H} -, \mathcal{J} -, and \mathcal{D} -class.

Using similar deductions, we can determine the structure of the remaining classes. For example there are an infinite number of single element \mathcal{L} -classes, there is exactly one two-element \mathcal{L} -class (which we described above), and there are three infinite \mathcal{L} -classes. Of these infinite \mathcal{L} -classes, two are characterized by “one loop” and are in the same \mathcal{D} -class. They are shown in Figure 6.6. There is also an infinite \mathcal{L} -class characterized by two loops. It is shown in Figure 6.7. It constitutes a $\mathcal{D} = \mathcal{J}$ -class as well. We give a partial eggbox diagram $\mathcal{M}(S, \{x, y\}, id)$ in Figure 6.5.

■



Figure 6.5: The eggbox diagram for $\mathcal{M}(S; \Sigma)$, which has infinite \mathcal{L} - and $\mathcal{D} = \mathcal{J}$ -classes. All $\mathcal{D} = \mathcal{J}$ -classes for elements of the form $(x, -, -)$ are shown, but none of the form $(y, -, -)$. The arrows between boxes indicate how elements are related under the \mathcal{R} relation. The element at the arrow's tail is greater under the \mathcal{R} relation than the element at the head. In the lower left, there is an \mathcal{R} -class with two elements that is also an \mathcal{L} -class. In the middle right, there is an \mathcal{R} -class with two elements that is not a \mathcal{L} -class. The \mathcal{H} -classes which are groups are indicated by the bold boxes.

“One Loop” \mathcal{D} -class

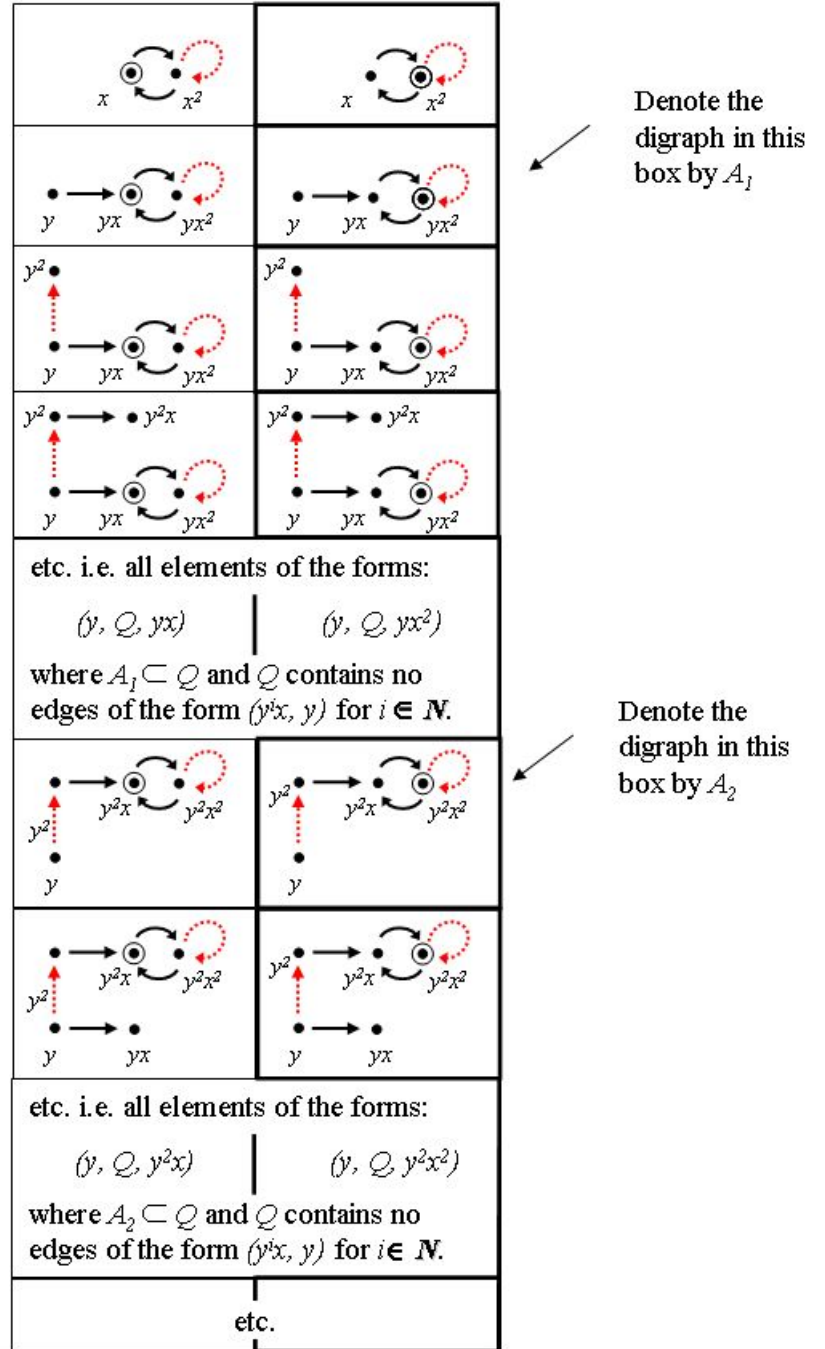


Figure 6.6: Eggbox diagram for the “One Loop” \mathcal{D} -class. The infinite \mathcal{D} -class for elements with one y -labeled loop is shown. This \mathcal{D} -class breaks down into two infinite \mathcal{L} -classes and an infinite number of two-element \mathcal{R} -classes. Its \mathcal{H} -classes all contain a single element. The \mathcal{H} -classes that are groups are indicated by the bold outline around their respective boxes.

“Two Loops” \mathcal{D} -class

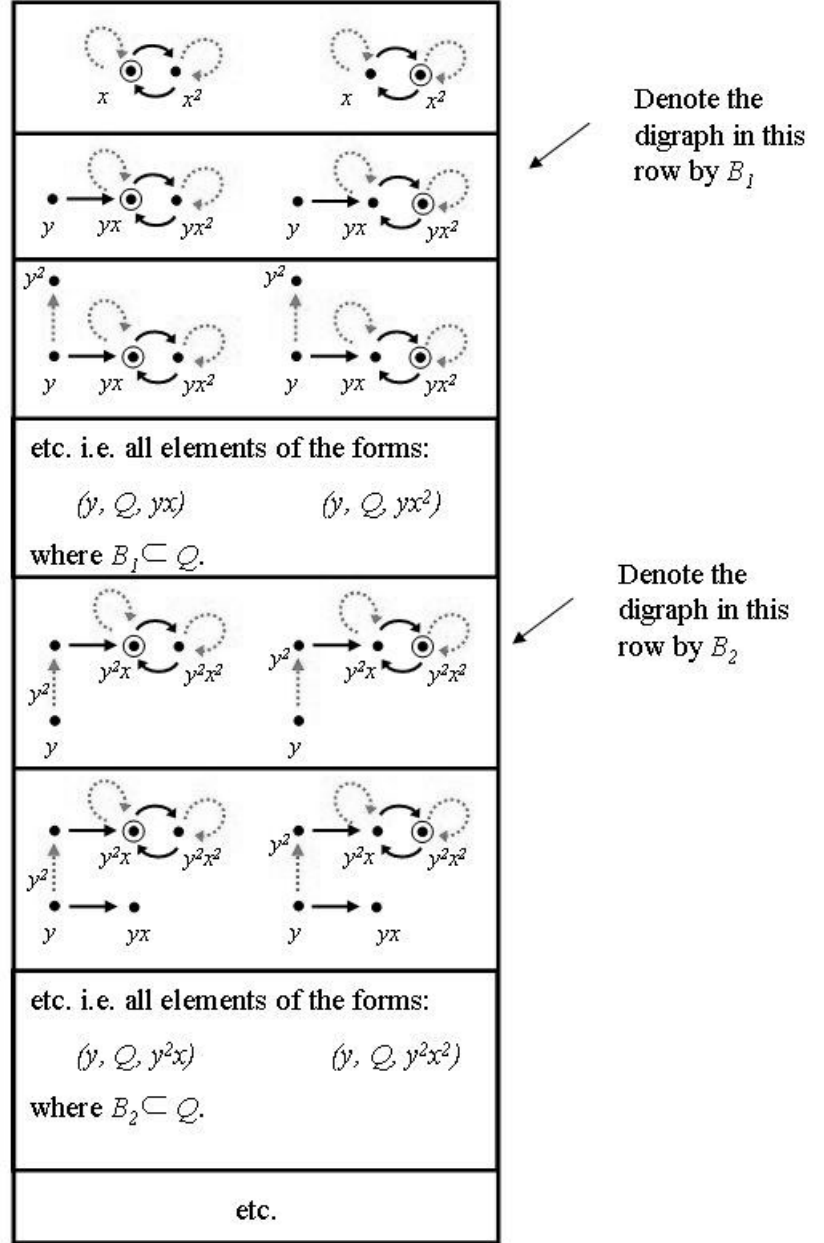


Figure 6.7: Eggbox diagram for the “Two Loops” \mathcal{D} -class. The infinite \mathcal{D} -class for elements with two y -labeled loops is shown. This \mathcal{D} -class is also an \mathcal{L} -class. Each row is a $\mathcal{R} = \mathcal{H}$ -class that is a group.

Chapter 7

Relationships Among Expansions

In this chapter we wish to compare the semigroup graph expansion with other expansions. We are interested in when homomorphisms between different types of expansions exist. To start, we will examine the connections between semigroup graph expansions and the Birget-Rhodes (semigroup) prefix expansion. Our main result in this section is that the semigroup path expansion plays a similar role with respect to the semigroup prefix expansion as the group graph expansion plays to the group prefix expansion.

In the second part of Chapter 7, we will consider the relationship between semigroup and monoid graph expansions with related inputs. We will show that the properties of the maps between them depend upon how the monoid and semigroup systems are related.

Throughout this chapter, identity elements will arise often. To avoid confusion, we will denote the identity of a group G by 1_G and of a monoid T by 1_T . If we add an identity to a semigroup graph expansion to form $\mathcal{M}(S; \Sigma)^1$, we will denote this identity by $1_{\mathcal{M}}$.

7.1 The Birget-Rhodes Prefix Expansion

Many expansions of groups and semigroups, with and without generating sets, are due to Birget and Rhodes (see [1] and [2]; Grillet also gives a clear description of many Rhodes' expansions in [8].) Of the various expansions, the right-prefix expansion and its cut-down-to-generators version are the most closely related to the graph expansions. In order to lay the framework for later generalizations, it is useful to see the right-prefix expansion and the right-prefix expansion cut-down-to-generators as subsemigroups of a larger expansion, which we will refer to as the subset expansion. If S is a semigroup, then the *semigroup subset expansion*, denoted \tilde{S}^{\subset} , is the set:

$$\{(X, x) \mid X \text{ is a finite subset of } S^1 \text{ and } x \in X\}$$

with multiplication defined by $(X, x)(Y, y) = (X \cup xY, xy)$ where $xY = \{xa \mid a \in Y\}$. Brief reflection reveals that the subset expansion is a semigroup.

We will call an expansion of S a *set-based expansion* if it is a subsemigroup of the subset expansion \tilde{S}^{\subset} . The *right-prefix expansion* of a semigroup S , denoted $\tilde{S}^{\mathcal{R}}$, is the following set-based expansion:

$$\left\{ (X, x) \left| \begin{array}{l} \text{there is a factorization } x = x_1x_2 \dots x_n \text{ with } x_i \in S \text{ such that} \\ X = \{1, x_1, x_1x_2, \dots, x_1x_2 \dots x_n\} \end{array} \right. \right\}.$$

Birget and Rhodes show that $\tilde{S}^{\mathcal{R}}$ is a semigroup, that it can be generated by the set $\{(\{1, s\}, s) \mid s \in S\}$, and that S is a homomorphic image of $\tilde{S}^{\mathcal{R}}$ via the projection $(X, x) \mapsto x$.

The right-prefix expansion can also be performed for groups by substituting a group G in the place of the semigroup S in the definition. This produces a monoid, denoted $\tilde{G}^{\mathcal{R}}$. Szendrei showed in [21] that the description of $\tilde{G}^{\mathcal{R}}$ can be expressed more simply:

$$\tilde{G}^{\mathcal{R}} = \{(X, x) \mid 1, x \in X\}.$$

We now look at set-based expansion whose input is a semigroup system, (S, Σ, f) . The *semigroup right-prefix expansion cut-down-to-generators*, denoted $\tilde{S}_\Sigma^\mathcal{R}$, is this subsemigroup of $\tilde{S}^\mathcal{R}$:

$$\left\{ (X, x) \in \tilde{S}^\mathcal{R} \left| \begin{array}{l} \text{there is a word } s_1 s_2 \dots s_n \text{ with } s_i \in \Sigma \text{ such that} \\ \{1, s_1 f, (s_1 s_2) f, \dots, (s_1 s_2 \dots s_n) f\} = X \text{ and} \\ (s_1 s_2 \dots s_n) f = x \end{array} \right. \right\}.$$

If we wish to determine the group analog, we will need to adjust for the fact that groups are generated differently than semigroups. If $grp(G, \Omega, f)$ is a group system, the *group right-prefix expansion cut-down-to-generators*, denoted $\tilde{G}_\Omega^\mathcal{R}$, is the subsemigroup:

$$\left\{ (X, x) \in \tilde{G}^\mathcal{R} \left| \begin{array}{l} \text{there is a word } s_1 s_2 \dots s_n \text{ with } s_i \in \Omega \cup \Omega^{-1} \text{ such that} \\ \{1, s_1 f, (s_1 s_2) f, \dots, (s_1 s_2 \dots s_n) f\} = X \text{ and} \\ (s_1 s_2 \dots s_n) f = x \end{array} \right. \right\}.$$

It is easy to construct a map from $M_{gp}(G; \Omega)$ to $\tilde{G}_\Omega^\mathcal{R}$. Suppose $(P, c) \in M_{gp}(G; \Omega)$. There is a word $w = s_1 s_2 \dots s_n$ (with $s_i \in \Omega \cup \Omega^{-1}$) which labels a $1 \rightarrow c$ path that traverses every edge of P . Thus we know that $(V(P), c) \in \tilde{G}_\Omega^\mathcal{R}$. The map $(P, c) \mapsto (V(P), c)$ is the desired map. Margolis and Meakin note in [17] that this map is a surjective homomorphism and that it is an isomorphism if and only if G is free on Ω .

In the next theorem, we show that the semigroup path expansion plays an analogous role with respect to the right-prefix expansion cut-down-to generators as the group graph expansion does to the right-prefix expansion. We will use the following definition: a semigroup system (S, Σ, f) is *left-cancellative on generators* if for any $r, s \in \Sigma$ and $x \in S$, the equation $x(rf) = x(sf)$ implies $r = s$. We observe that if (S, Σ, f) is left-cancellative on generators, then the map $f : \Sigma \rightarrow S$ is injective.

Theorem 7.1.1. *Let (S, Σ, f) be a semigroup system of a semigroup S . Then the map $\eta : \text{Path}(S; \Sigma)^1 \rightarrow \tilde{S}_\Sigma^{\mathcal{R}}$ defined by*

$$\begin{aligned}\eta : \quad (r, P, c) &\mapsto (\{1\} \cup V(P), c) \\ 1_{\mathcal{M}} &\mapsto (\{1\}, 1)\end{aligned}$$

is a surjective homomorphism. It is injective if and only if the system (S, Σ, f) is left-cancellative on generators and for all $x \in S$, we have that $x \notin xS$.

Proof: We first prove that η is a surjective homomorphism. Let $(r, P, c), (s, Q, d) \in \text{Path}(S; \Sigma)$. Then

$$\begin{aligned}(r, P, c)\eta(s, Q, d)\eta &= (\{1\} \cup V(P), c)(\{1\} \cup V(Q), d) \\ &= (\{1\} \cup V(P) \cup \{c\} \cup cV(Q), cd) \\ &= (\{1\} \cup V(P \cup cQ_s^1), cd) \\ &= (r, P \cup cQ_s^1, cd)\eta \\ &= ((r, P, c)(s, Q, d))\eta.\end{aligned}$$

We now show that η is surjective. Let $(X, c) \in \tilde{S}_\Sigma^{\mathcal{R}}$. Then there exists a word $w \in \Sigma^+$ with $wf = c$ such that $X = \{yf \mid y \text{ is a prefix of } w\}$. Write w as sv where $s \in \Sigma$ and $v \in \Sigma^*$. Since $(s, [sf \xrightarrow{v} c], c)\eta = (X, c)$, we see that η is surjective.

Suppose η is injective. Let $x \in S$. We will first show that $x \notin xS$. By way of contradiction, we suppose the set $Y = \{x \mid x \in xS\}$ is non-empty. Next, let $\Upsilon = \{w \mid wf = y \text{ for some } y \in Y\}$. By the well ordering principle, we can find a $w_y \in \Upsilon$ corresponding to some $y \in Y$ such that $|w_y| \leq |w|$ for any $w \in \Upsilon$. We rewrite w_y as $w_y = r_y v_y$ where $r_y \in \Sigma$ and $v_y \in \Sigma^*$. Since $y \in Y$, there also exists a word $u \in \Lambda^+$ that labels a cycle based at the vertex y . We can choose u so that the path $y \xrightarrow{u} y$ intersects itself only at y . The paths $r_y \xrightarrow{v_y} y$ and $y \xrightarrow{u} y$ have no edges in common, because if they did, we could find an element $b \in Y$ and a word $w_b \in \Upsilon$ for

which $w_b < w_y$.

We rewrite u as $u = zt$ where $z \in \Sigma^*$ and $t \in \Sigma$. Consider the digraphs: $P = [r_y \xrightarrow{v_y z} y(zf)]$ and $Q = [r_y \xrightarrow{v_y u} y(zf)]$. They are different, since Q contains the edge $(y(zf), t)$, but P does not. However, as their vertex sets are the same, $(r_y, P, y(zf))\eta = (r_y, Q, y(zf))\eta$, contradicting the fact that η is injective. We conclude that $x \notin xS$ for all $x \in S$.

We now wish to show that (S, Σ, f) is left-cancellative on generators. To this end, let $s, t \in \Sigma$ and $x \in S$ be such that $x(sf) = x(tf)$. Choose a word $w \in \Sigma^+$ such that $wf = x$. As usual, rewrite w as $w = rv$ where $r \in \Sigma$ and $v \in \Sigma^*$. Using the result that $x \notin xS$ for all $x \in S$, we know there is no prefix w' of w such that $w'f = x$. Thus the path $rf \xrightarrow{v} x$ does not contain the edges (x, s) or (x, t) . However, we have that $(r, [rf \xrightarrow{vs} x(sf)], x(sf))\eta = (r, [rf \xrightarrow{vt} x(sf)], x(sf))\eta$. Since η is injective, $[rf \xrightarrow{vs} x(sf)] = [rf \xrightarrow{vt} x(sf)]$, from which it follows that $s = t$.

Conversely, assume that for all $x \in S$ we have that $x \notin xS$ and that the system (S, Σ, f) is left-cancellative on generators. Suppose $(r, P, c), (s, Q, d) \in \text{Path}(S; \Sigma)$ are such that $(V(P), c) = (r, P, c)\eta = (s, Q, d)\eta = (V(Q), d)$. Consider the case when $|V(P)| = 1$. Since $V(P) = V(Q)$, we have that $rf = sf$. It follows from this that $(rf)(rf) = (rf)(sf)$, from which we see that $r = s$. Moreover, neither P nor Q can contain edges, since there are no elements y such that $(rf)y = rf$. Thus $(r, P, c) = (s, Q, d)$.

Now consider the case when $|V(P)| \geq 2$. There are words $v_1 v_2 \dots v_m, w_1 w_2 \dots w_n$ with each $v_i, w_j \in \Sigma$ such that $P = [rf \xrightarrow{v_1 v_2 \dots v_m} c]$ and $Q = [sf \xrightarrow{w_1 w_2 \dots w_n} c]$. Since there are no cycles, the paths $rf \xrightarrow{v_1 v_2 \dots v_m} c$ and $sf \xrightarrow{w_1 w_2 \dots w_n} c$ pass through vertices only once and they visit these vertices in the same order. This implies that $rf = sf$, whereupon $r = s$. Moreover, $(rf)(v_1) = (rf)(w_1)$, from which it follows that $v_1 = w_1$. An inductive argument then shows that $v_i = w_i$ for $1 \leq i \leq m$ and that $m = n$. Therefore we have that $(r, P, c) = (s, Q, d)$ in this case as well, whereupon we conclude that η is injective. ■

In contrast, except for the instances when $\mathcal{M}(S; \Sigma) = \text{Path}(S; \Sigma)$, we can not use this method to construct a surjective homomorphism between $\mathcal{M}(S; \Sigma)$ and $\tilde{S}_\Sigma^\mathcal{R}$. This motivates us to define a new set-based expansion that for which we can generalize this method. Define the *right-factor expansion cut-down-to-generators*, denoted $\tilde{S}_\Sigma^\mathcal{F}$, to be the following subsemigroup of $\tilde{S}^\mathcal{C}$:

$$\left\{ (X, x) \left| \begin{array}{l} \text{there exists an } r \in \Sigma \text{ such that } rf \in X \text{ and for every } v \in X, \\ \text{there is a word } v_1 v_2 \dots v_n \in \Sigma^* \text{ such that } v = (rv_1 v_2 \dots v_n)f \text{ and} \\ \{(rv_1)f, (rv_1 v_2)f, \dots (rv_1 v_2 \dots v_{n-1})f\} \subseteq X \end{array} \right. \right\}.$$

Note that if $(X, x) \in \tilde{S}_\Sigma^\mathcal{F}$, then the set X is finite since $\tilde{S}_\Sigma^\mathcal{F} \subseteq \tilde{S}^\mathcal{C}$. It is easy to see from the definition that $\tilde{S}_\Sigma^\mathcal{R}$ is a subsemigroup $\tilde{S}_\Sigma^\mathcal{F}$. Moreover, we can extend the map $\eta : \text{Path}(S; \Sigma)^1 \rightarrow \tilde{S}_\Sigma^\mathcal{R}$ to $\mathcal{M}(S; \Sigma)^1$ and show that its image is $\tilde{S}_\Sigma^\mathcal{F}$.

Proposition 7.1.2. *Let (S, Σ, f) be a semigroup system. Then the map $\eta : \mathcal{M}(S; \Sigma)^1 \rightarrow \tilde{S}^\mathcal{F}$ defined by:*

$$\eta : (r, P, c) \mapsto (\{1\} \cup V(P), c)$$

$$1 \mapsto (\{1\}, 1)$$

is a surjective homomorphism.

Proof: The proof that η is a homomorphism is the same as in Proposition 7.1.1 and we omit it. We now show that η is surjective. Let $(X, x) \in \tilde{S}_\Sigma^\mathcal{F}$. Then there exists some $r \in \Sigma$ such that to every element $y \in X$, there corresponds a word rw_y such that $(rw_y)f = y$ and if v is a prefix of rw_y , then $vf \in X$. We construct the following subdigraph of $\text{Cay}(S; \Sigma)$:

$$P = \bigcup_{y \in X} [rf \xrightarrow{w_y} y].$$

Note that P is a finite subdigraph, since it is the union of a finite number of finite

paths. It is rooted at rf and contains the vertex x . Thus, $(r, P, x) \in \mathcal{M}(S; \Sigma)$ and $(r, P, x)\eta = (X, x)$. ■

We summarize the connections between the expansions in Theorems 7.1.1 and 7.1.2 in the following diagram.

$$\begin{array}{ccc}
 \text{Path}(S; \Sigma)^I & \leq & \mathcal{M}(S; \Sigma)^I \\
 \eta \downarrow & & \downarrow \eta \\
 \widetilde{S}_\Sigma^R & \leq & \widetilde{S}_\Sigma^F
 \end{array}$$

7.2 Monoid and Semigroup Graph Expansions

In order to explore the relationship between monoid and semigroup graph expansions, we will start by forming semigroup systems from monoid systems and then investigate the maps from the respective semigroup graph expansions to the monoid graph expansions. The properties of these maps depend on how we construct the systems. We discuss three different scenarios: first, forgetting the identity in the semigroup system; second, adding an identity to the semigroup graph expansion; third, adding a generator to the semigroup system to ensure that the identity is generated.

Proposition 7.2.1. *Consider a monoid system $\text{mon}(T, \Lambda, f)$ and form a semigroup system $\text{sgp}(S, \Lambda, f)$ where S is the subsemigroup of T that is generated (as a semigroup) by Λ . Let $\gamma : \mathcal{M}(S; \Lambda) \rightarrow M_{\text{mon}}(T; \Lambda)$ be the map defined by $(r, P, c) \mapsto (P_r^1, c)$. Then:*

- (a) γ is a homomorphism;
- (b) γ is not surjective;
- (c) γ is injective if and only if $S \neq T$.

Proof: Starting with part (a), let $(r, P, c), (s, Q, d) \in \mathcal{M}(T; \Lambda)$. Then

$$\begin{aligned}
((r, P, c)(s, Q, d))\gamma &= (r, P \cup cQ_s^1, cd)\gamma \\
&= ((P \cup cQ_s^1)_r^1, cd) \\
&= (P_r^1 \cup cQ_s^1, cd) \\
&= (P_r^1, c)(Q_s^1, d) \\
&= (r, P, c)\gamma(s, Q, d)\gamma.
\end{aligned}$$

We conclude that γ is a homomorphism.

To see why (b) is true, observe that all elements in $\mathcal{M}(S; \Lambda)$ are mapped by γ to elements that have digraphs with edges. Thus no element maps to $(\bullet_{1_T}, 1_T)$.

We now wish to show (c). Suppose γ is injective. By way of contradiction, assume $S = T$. Then there exists some minimal length word $w \in \Sigma^+$ such that $wf = 1_T$. Rewrite w as rv where $r \in \Sigma$ and $v \in \Sigma^*$. Let $P = \lfloor rf \xrightarrow{v} 1_T \rfloor$. Note that P does not contain the edge $(1_T, r)$ because we assumed that w had minimal length. Construct a new graph P' with $V(P') = V(P)$ and $E(P') = E(P) \cup \{(1_T, r)\}$. Both P and P' are rf -rooted digraphs of $\text{Cay}_{sgp}(S; \Lambda)$. Moreover, $(r, P, 1_T)\gamma = (r, P', 1_T)\gamma$. This contradicts the assumption that γ is injective. We conclude that $S \neq T$.

Suppose $S \neq T$. This implies that $1_T \notin S$. Let $(r, P, c)\gamma = (s, Q, d)\gamma$. Thus $P_r^1 = Q_s^1$ and $c = d$. Since $1_T \notin S$, we know that $1_T \notin V(P), V(Q)$. It follows that $(1_T, s) = (1_T, r)$, whereupon we have that $r = s$ and $P = Q$. We conclude that γ is injective. ■

Proposition 7.2.2. *Consider a monoid system $\text{mon}(T, \Lambda, f)$ and form a semigroup system $\text{sgp}(S, \Lambda, f)$ where S is a subsemigroup of T generated (as a semigroup) by Λ . Let $\gamma : \mathcal{M}(S; \Lambda)^1 \rightarrow M_{\text{mon}}(T; \Lambda)$ be the map defined by $1_{\mathcal{M}} \mapsto (\bullet, 1_T)$ and $(r, P, c) \mapsto (P_r^1, c)$. Then:*

- (a) γ is a homomorphism;
- (b) γ is surjective if and only if T is the trivial group;
- (c) γ is injective if and only if $S \neq T$.

Proof:

Proving part (a) and (c) requires extending the proofs in Proposition 7.2.1(a) and (c) to also apply to the element $1_{\mathcal{M}}$. These are easy calculations and we omit them.

In order to show (b), we first assume that T is the trivial group. Let $(P, 1_T) \in M_{\text{mon}}(T; \Lambda)$. If $(P, 1_T) = (\bullet, 1_T)$, then $1_{\mathcal{M}}\gamma = (P, 1_T)$. If $(P, 1_T) \neq (\bullet, 1_T)$, then there is some $r \in \Sigma$ such that $(1_T, r) \in E(P)$. Because T is the trivial group, we have that $rf = 1_T$. Thus $(r, P, 1_T)\gamma = (P, 1_T)$. We see that γ is surjective.

For the converse, suppose γ is surjective. By way of contradiction, assume T is not trivial. Then there exists some $r \in \Lambda$ such that $rf \neq 1_T$. Consider the element $(\bullet \xrightarrow{1_T} \bullet, 1_T) \in M_{\text{mon}}(T; \Lambda)$. Suppose $(s, P, 1_T)\gamma = (\bullet \xrightarrow{1_T} \bullet, 1_T)$. If $sf = 1$, P_s^1 contains an sf -labeled loop at 1_T . If $sf \neq 1$, there exists some $sf \rightarrow 1_T$ path in P . These are both contradictions. We conclude that T is trivial. ■

For this next scenario, we need two techniques, loop deletion and loop addition. Let Γ be a deterministic digraph and let $t \in \Sigma(\Gamma)$ be such that Γ contains a t -labeled loop at each vertex. Furthermore, let $P \subseteq \Gamma$. Then the digraph P after t -deletion, denoted P^i , is the maximal subdigraph of P containing no t -labeled loops. The digraph P after t -addition, denoted by P^+ , is the digraph obtained from P by including all t -labeled loops at vertices in P .

Lemma 7.2.3. *Let $\text{sgp}(S, \Sigma, f)$ be a semigroup system. If $P, Q \subseteq \text{Cay}_{\text{sgp}}(S, \Sigma)$, $c \in S$, and $t \in \Sigma \cup \Sigma^{-1}$ is such that tf is a right identity of S , then we have the following:*

- (a) $(P \cup Q)^t = P^t \cup Q^t$;
- (b) $(cP)^t = c(P^t)$;
- (c) $(P \cup Q)^{\dagger} = P^{\dagger} \cup Q^{\dagger}$;
- (d) $(cP)^{\dagger} = c(P^{\dagger})$.

Proof: Parts (a) and (c) follow from the fact that $V(P \cup Q) = V(P) \cup v(Q)$. We obtain (b) from the fact that deleting t -labeled loops does not affect edges with labels other than t . Similarly, adding t -labeled loops does not affect the other edges and thus part (d) follows. ■

We will also use the concept of retracts. Recall that a retract of a semigroup S is subsemigroup X for which there is an endomorphism from S onto X that is the identity map when restricted to X .

Proposition 7.2.4. *Consider a monoid system $\text{mon}(T, \Lambda, f)$ and create a semigroup system $\text{sgp}(T, \Lambda \cup \{e\}, g)$ where the map g agrees with f on Λ and $eg = 1_T$. Let $\gamma : \mathcal{M}(T, \Lambda \cup \{e\}) \rightarrow M_{\text{mon}}(T; \Lambda)$ be the map defined by $(r, P, c) \mapsto ((P_r^1)^e, c)$. Then:*

- (a) γ is a homomorphism;
- (b) γ is surjective;
- (c) $M_{\text{mon}}(T; \Lambda)$ is isomorphic to a retract of $\mathcal{M}(T, \Lambda \cup \{e\})$.

Proof: Beginning with part (a), let $(r, P, c), (s, Q, d) \in \mathcal{M}(T, \Lambda \cup \{e\})$. Using

Lemma 7.2.3, we have that:

$$\begin{aligned}
((r, P, c)(s, Q, d))\gamma &= (r, P \cup cQ_s^1, cd)\gamma \\
&= \left(((P \cup cQ_s^1)_r^1)^\dot{e}, cd \right) \\
&= \left((P_r^1 \cup cQ_s^1)^\dot{e}, cd \right) \\
&= \left((P_r^1)^\dot{e} \cup c((Q_s^1)^\dot{e}), cd \right) \\
&= ((P_r^1)^\dot{e}, c)((Q_s^1)^\dot{e}, d) \\
&= (r, P, c)\gamma(s, Q, d)\gamma.
\end{aligned}$$

Thus γ is a homomorphism.

Moving on to part (b), we consider an element $(P, c) \in M_{mon}(T; \Lambda)$. Note that since $1 = ef$, the digraph P is ef -rooted, but contains no e -labeled loops, since it is a subdigraph of the monoid Cayley digraph. By adding and then removing the loop $(1_T, e)$, it follows that $(P_e^1)^\dot{e} = P$. Thus $(e, P, c)\gamma = ((P_e^1)^\dot{e}, c) = (P, c)$, whereupon γ is surjective.

Finally, we wish to show part (c). Consider the following subset of $\mathcal{M}(T, \Lambda \cup \{e\})$:

$$X = \{(e, P, c) \mid \text{if } v \in V(P), \text{ then } (v, e) \in E(P)\}$$

It is easy to see that X is a submonoid of $\mathcal{M}(T, \Lambda \cup \{e\})$ with identity element $(e, A, 1_T)$, where A is the digraph consisting of the vertex 1_T and the e -labeled loop at 1_T . It is not difficult to see that γ is injective on X and has image $M_{mon}(T; \Lambda)$.

Let $\beta : \mathcal{M}(T, \Lambda \cup \{e\}) \rightarrow \mathcal{M}(T, \Lambda \cup \{e\})$ be the map given by $(r, P, c) \mapsto (r, P_e^\dagger, c)$. Calculations similar to those used in part (a) show that β is a homomorphism. Moreover, it has image X and is injective when restricted to X . Thus X is a retract of $\mathcal{M}(T, \Lambda \cup \{e\})$, whereupon $M_{mon}(T; \Lambda)$ is isomorphic to a retract of $\mathcal{M}(T, \Lambda \cup \{e\})$. ■

Monoid Graph Expansions and Semigroup Path Expansions

Propositions 7.2.1, 7.2.2, and 7.2.4 all hold if we replace the graph expansion $\mathcal{M}(S; \Sigma)$ by $\text{Path}_{sgp}(S; \Sigma)$ and $M_{mon}(T; \Lambda)$ by $\text{Path}_{mon}(T; \Lambda)$. Additionally, we obtain a new Corollary to Proposition 7.2.2:

Corollary 7.2.5. *Consider a monoid system $mon(T, \Lambda, f)$ and form a semigroup system $sgp(S, \Lambda, f)$ where S is a subsemigroup of T generated (as a semigroup) by Λ . If $S \neq T$, then $(\text{Path}_{sgp}(S; \Lambda))^1 \cong \text{Path}_{mon}(T; \Lambda)$.*

Proof: Suppose $S \neq T$. We use the function γ described in Proposition 7.2.2 and restrict it to $(\text{Path}_{sgp}(S; \Lambda))^1$. From Proposition 7.2.2(a) and (c), γ is an injective homomorphism. We now show that it is surjective. Let $(P, c) \in \text{Path}_{mon}(T; \Lambda)$. If $(P, c) = (\bullet_{1_T}, 1_T)$, then $1_{\mathcal{M}}\gamma = (P, c)$. If $(P, c) \neq (\bullet_{1_T}, 1_T)$, there exists some word $w \in \Lambda^+$ such that w labels a path $1 \xrightarrow{w} c$. Rewrite w as $w = rv$ where $r \in \Lambda$ and $v \in \Lambda^*$. Then $(r, [rf \xrightarrow{v} c], c) \in \text{Path}_{sgp}(S; \Lambda)$ and $(r, [rf \xrightarrow{v} c], c)\gamma = (P, c)$, whereupon we have that γ is surjective. We conclude that γ is an isomorphism and hence $(\text{Path}_{sgp}(S; \Lambda))^1 \cong \text{Path}_{mon}(T; \Lambda)$. ■

Chapter 8

Remaining Questions

In this final section, we would like to mention some unanswered questions about the semigroup graph and path expansions. We will also include our observations relating to these questions.

First, we would like to know more about the basic structure of the semigroup graph expansion. For example, it would be interesting to see if it embeds in a semigroup which is a direct or semidirect product of simpler semigroups. We have looked at the semigroup $\Sigma \times \left(\mathcal{P}(\text{Cay}_{\text{mon}}(S^1; \Sigma)) \rtimes S \right)$ where the operation on Σ is left-zero multiplication, the operation on $\mathcal{P}(\text{Cay}_{\text{mon}}(S^1; \Sigma))$ is set union, S keeps its original operation, and the action of each $c \in S$ on each $P \in \mathcal{P}(\text{Cay}_{\text{mon}}(S^1; \Sigma))$ is given by cP . This did yield a partial result:

Proposition 8.0.6. *Let (S, Σ, f) be a semigroup system of a semigroup that is not a monoid. Then $\mathcal{M}(S; \Sigma)$ embeds in $\Sigma \times \left(\mathcal{P}(\text{Cay}_{\text{mon}}(S^1; \Sigma)) \rtimes S \right)$ via the homomorphism $\gamma : \mathcal{M}(S; \Sigma) \rightarrow \Sigma \times \left(\mathcal{P}(\text{Cay}_{\text{mon}}(S^1; \Sigma)) \rtimes S \right)$ defined by $(r, P, c) \mapsto (r, (P_r^1, c))$.*

Proof: We first show that γ is a homomorphism. Let $(r, P, c), (s, Q, d) \in \mathcal{M}(S; \Sigma)$.

Then we have:

$$\begin{aligned}
(r, P, c)\gamma(s, Q, d)\gamma &= (r, (P_r^1, c))(s, (Q_s^1, d)) \\
&= (r, (P_r^1 \cup cQ_s^1, cd))\gamma \\
&= (r, P \cup cQ_s^1, cd)\gamma \\
&= ((r, P, c)(s, Q, d)).
\end{aligned}$$

Moreover if $(r, P, c)\gamma = (s, Q, d)\gamma$, then $P_r^1 = Q_s^1$ and $c = d$. Since S is not a monoid, neither P nor Q contain the edge $(1, r)$. Thus we have that $(1, r) = (1, s)$, whereupon $r = s$. We conclude that γ is injective and hence an embedding. \blacksquare

We tried to modify this idea to see if we could embed $\mathcal{M}(S; \Sigma)$ in the semigroup $\Sigma \times \left((\mathcal{P}(\text{Cay}_{\text{mon}}(S^1; \Sigma)) \rtimes S \right)$, but we were unable to do so. The difficulty occurs for elements (r, P, c) , where P contains a $[rf \rightarrow 1]$ path, since the image of (r, P, c) and (r, P_r^1, c) under γ are the same. Alternatively, we tried to embed $\mathcal{M}(S; \Sigma)$ in $\Sigma \times \left(\mathcal{P}(\text{Cay}(S; \Sigma)) \rtimes S \right)$, but we were not able to capture the edge insertion that occurs in the product $(r, P, c)(s, Q, d) = (r, P \cup cQ_s^1, cd)$ (by edge insertion we are referring to the inclusion of the edge (c, s)). Thus, finding a decomposition of the semigroup graph expansion remains an open problem. Similarly, the structural decomposition of the semigroup path expansion is also unknown.

A second remaining question is whether we can characterize the semigroup graph and path expansions as specific objects in appropriate categories. This has been done for the other graph expansions: for example Margolis and Meakin show that the group graph expansion $M_{gp}(G; \Omega)$ is the initial object in the category of Ω -generated inverse semigroups with maximal group image G (see [17]) and Elston determines that the semigroup Cayley expansion $\text{CayExp}(S; \Sigma)$ is the largest expansion for which the local semigroups of the derived category are semilattices (see [3]). Finding an analogous characterization for the semigroup graph and path expansions remains.

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